Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 20 problems that are to be completed in 60 minutes.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 20 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.
- All answers are of one of the following forms. Please follow these instructions for converting your answer into an integer, so all answers submitted to the Google form are integers.
  - a for an integer a
    - Submit a
  - $\frac{a}{b}$ for relatively prime positive integers a, b with $b > 1$
    - Submit $a + b$
  - $a\pi + b$ for integers a, b with $a \neq 0$
    - Submit $a + b$
  - $\sqrt{a}$ for a positive nonsquare positive integer a
    - Submit a
  - $\frac{a\sqrt{b}}{c}$ for a squarefree positive integer b, positive integer $c > 1$, and $\gcd(a, c) = 1$
    - Submit $a + b + c$

1. Sara writes the equation
$$35 + 6z = 71$$
on a whiteboard for Zoe to solve. Since Zoe isn't wearing her glasses, the 5 looks like an s and the z looks like a 2. What is the positive difference between Sara's answer for z and Zoe's answer for s?

2. Toto has 10 boxes each containing 5 beans and 5 boxes each containing 10 beans. Each box contains one pinto bean. If Toto selects a box at random, and then selects a bean from that box at random, what is the probability that he selects a pinto bean?

3. In triangle GDL, I and E lie on sides GD, GL respectively such that $GI = 5$, $ID = 11$, $EG = 8$, $EL = 2$, and $IE = 7$. What is DL?

4. Compute
$$\frac{1}{\frac{1}{41+9} + \frac{1}{41-9}} + \frac{1}{\frac{1}{41+40} + \frac{1}{41-40}}.$$

5. Mamamoo is a K-pop girl group consisting of the members Solar, Moonbyul, Wheein, and Hwasa. In their song "Wind Flower", the members sing one at a time for a total of 4 minutes of singing. Moonbyul sings for a third of amount of time that Wheein and Hwasa collectively sing. Given that Moonbyul sings for the least amount of time, what is the greatest possible difference between the number of seconds that Wheein and Hwasa sing?

6. How many ways are there to arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8 in a circle such that no two adjacent numbers share a common factor greater than 1? Two arrangements are considered to be the same if they are rotations of each other.

7. A class of students took a test, each receiving an integer grade from 0 to 10 inclusive. If the average score of the top 25 students was $9\frac{1}{5}$, and the average score of all the students was $6\frac{2}{7}$, what is the smallest possible number of students?

8. You and five of your friends line up in a random order to get tickets, one at a time. There are sixteen tickets available, labelled 1 through 16. Your friends do not care which ticket they get, and will take one at random; but your favorite number is 13, so if ticket #13 is available, you will take it. What is the probability that you get ticket #13?

9. An acute triangle and a circle centered at the triangle's incenter both have an area of 4π. Given that $\frac{3}{4}$ of the circle's circumference lies outside the triangle, find the total area of the regions that are inside the triangle and outside the circle.

10. Let $ABCD$ be a rectangle, and let G be the centroid of ABC. Given that $DA = DG$, find $\frac{AD}{CD}$.

11. Call a positive integer *redundant* if all of its digits appear at least twice in its base-10 representation. For example, 1102021 is redundant, but 11223 is not. Given that 2020 is the nth smallest redundant number, find n.

12. A rectangular garden is built, and a rectangular walkway with fixed width is built around it. If the rectangles formed by the garden and walkway both have integral width, sides, and diagonals, what is the smallest possible area of the walkway?

13. Define a sequence such that $a_1 = 2$, and for $n \geq 2$, a_n is the smallest possible positive integer greater than a_{n-1} such that there do not exist $1 \leq i, j \leq n$, not necessarily distinct, with $a_i + a_j = a_n$. Compute a_{2020}.

14. Let x and y be real numbers such that $\cos x \neq 0$ and $\tan x = 3^{y+1}$. What is the largest possible value of $\sec x - 9^y$?

15. Serena tells Thomas and Ulysses that she is thinking of a two-digit positive integer. She truthfully and separately tells Thomas the tens digit, from which he can conclude that her number is not a perfect square, and Ulysses the units digit, from which he can also conclude that her number is not a perfect square. What is the sum of all possible numbers Serena is thinking of?

16. Alice flips a fair coin until she flips her third "tails", at which point she stops. Given that she flips the coin at least five times and that the fifth flip was "heads," find the expected number of times that Alice flips the coin.

17. There is a unique increasing sequence $p_1, p_2, \ldots p_k$ of primes for $k \geq 2$ such that $p_k = 13$ and

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} - \frac{1}{p_1 p_2 \cdots p_k}$$

is an integer. Compute $p_1 + p_2 + \cdots + p_k$.

18. Let $ABCD$ be a convex quadrilateral with $AB = 6$, $BC = 9$, $CD = 15$, and $DA = 10$. Point P is chosen on line AC so that $PB = PD$. Given that $AC = 11$, find PB.

19. The polynomials

$$P(x) = x^3 - 7x^2 + ax + b$$

and

$$Q(x) = x^5 - 5x^4 + cx^3 + dx^2 + ex + f.$$

have the same set of roots, where a, b, c, d, e, f are real numbers. If $P(3) = 6$, find the sum of all possible values for $Q(3)$.

20. Let ABC be a triangle, and let the excenter opposite A be I_A. Points P, Q lie on the extensions of sides AB, AC past B, C respectively so that $BP = BC = CQ$. Suppose P, I_A, Q are collinear. Given that $AB = 20$ and $AC = 21$, find BC.

Team Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 12 problems that are to be completed in 30 minutes, along with a small minigame that will generate a multiplier for a team's team round score.
- There is no partial credit or penalty for incorrect answers.
- The minigame, or problem 13, will ask your team to submit a rational number, and will generate a multiplier between 0.9 and 1.1 for your team round score. Be sure to try out the minigame, as submitting nothing will decrease your team round score.
- Each of the 12 team round problems have a predetermined point value, indicated next to the problem numbers; your team's team round score will be the sum of the point values assigned to each question that is correctly answered, multiplied by the multiplier generated by the minigame. Excluding the multiplier, a perfect score (achieved through answering all 12 problems correctly) is **400 points**.
- Your team score will be a combination of your score on the team round and the scores of each individual member.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your team name, team members' names, and all answers on your answer form. Only the responses on the answer forms will be graded.
- All answers are of one of the following forms. Please follow these instructions for converting your answer into an integer, so all answers submitted to the Google form are integers.
  - a for an integer a
    - Submit a
  - $\frac{a}{b}$ for relatively prime positive integers a, b with $b > 1$
    - Submit $a + b$
  - $a\pi + b$ for integers a, b with $a \neq 0$
    - Submit $a + b$
  - $\sqrt{a}$ for a positive nonsquare positive integer a
    - Submit a
  - $\frac{a\sqrt{b}}{c}$ for a squarefree positive integer b, positive integer $c > 1$, and $\gcd(a, c) = 1$
    - Submit $a + b + c$

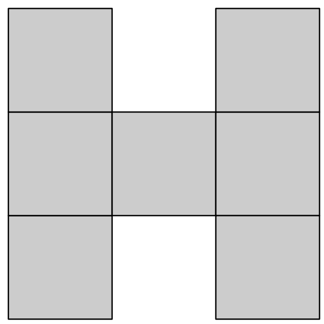
1. **[22]** Let c be a real number. Suppose that the line with equation $y = cx$ is perpendicular to the line with equation $x + \frac{y}{8} = 2cx + 1$. What is the value of c?

2. **[24]** Two unit circles are given such that their common internal tangent has half the length of their common external tangent. Find the distance between their centers.

3. **[26]** Ethan has 5 different books: one yellow, two green, and two red. If the yellow book is between two books of the same color (but not necessarily adjacent to these two books), how many ways can he arrange the books on his shelf?

4. **[28]** The cells of a 12×12 multiplication table are filled in such that the cell in the ith row and jth column is labeled with the product $i \cdot j$. How many of these products end with a 0?

5. **[30]** Holden makes the letter H, a connected figure with seven blocks, as depicted. He then removes blocks one at a time without ever disconnecting the figure until all blocks are removed. Two blocks that only touch at a corner are not considered connected, so for example, Holden cannot remove the middle block or the blocks to its left or right on the first turn. How many different ways can he do this?



6. **[32]** Let $ABCD$ be a rectangle with diagonals $AC = BD = 12$. Similar isosceles triangles EAB, FBC, GCD, HDA with bases AB, BC, CD, DA respectively are drawn outside the rectangle and have total area 24. If the areas of $ABCD$ and $EFGH$ are equal, find this common area.

7. **[34]** A sequence of six prime numbers is given such that

   - the first term has two digits, and
   - each term is one more than twice its previous term.

   What is the last term?

8. **[36]** A polygon with all angles equal to $90°$ or $270°$ has the property that its sides have distinct prime lengths. Find the minimum possible perimeter.

9. **[38]** Let x, y, z be nonzero complex numbers such that $x + y + z = 5$, $x^2y + y^2z + z^2x = 20$, and $xy^2 + yz^2 + zx^2 = 19$. Compute
$$\frac{x^3 + y^3 + z^3 - 8}{xyz}.$$

10. **[40]** Let P be a point on the incircle ω of unit equilateral triangle ABC. Let X be the second intersection of ω and $\overline{AP}$ and Y be the second intersection of ω and $\overline{BP}$. Find the maximum possible area of triangle PXY.

11. **[42]** Let α be a real number satisfying $\alpha^3 = \alpha + 1$. Find $m^2 + n^2$ if $m < n$ are integers satisfying $\alpha^m + \alpha^n = 2$.

12. **[44]** Find the smallest positive integer with the property that the product of its prime divisors is equal to the sum of its nonprime proper divisors (a proper divisor is a positive integer divisor of a number, excluding itself).

13. Let X be the fraction of teams that will submit an answer less than $\frac{1}{2}$ to this problem. Estimate X. Your multiplier will be $0.9 + 0.2 \times \left(\min\left\{\frac{X}{Y}, \frac{Y}{X}\right\}\right)^2$, where Y is the answer you submit. If you submit nothing, an irrational number, or a number outside the range $(0, 1]$, your multiplier will default to 0.9.

Answers

Individual Round

1. 3
2. 7 $\left(\frac{1}{6}\right)$
3. 14
4. 43 $\left(\frac{41}{2}\right)$
5. 48
6. 72
7. 42
8. 59 $\left(\frac{27}{32}\right)$
9. -3 $(3\pi - 6)$
10. 12 $\left(\frac{2\sqrt{5}}{5}\right)$
11. 48
12. 60
13. 5048
14. 121 $\left(\frac{85}{36}\right)$
15. 1110
16. 100 $\left(\frac{89}{11}\right)$
17. 29
18. 71 $\left(\frac{66}{5}\right)$
19. 90 (96 and -6)
20. 421 $(\sqrt{421})$

Team Round

1. $5 \left(\frac{1}{4}\right)$
2. $10 \left(\frac{4\sqrt{3}}{3}\right)$
3. 64
4. 33
5. 240
6. 54
7. 2879
8. 98
9. -6
10. $20 \left(\frac{1\sqrt{3}}{16}\right)$
11. 29 $(m = -5, n = 2)$
12. 42

Solutions

Individual Round

1. Sara writes the equation
$$35 + 6z = 71$$
on a whiteboard for Zoe to solve. Since Zoe isn't wearing her glasses, the 5 looks like an s and the z looks like a 2. What is the positive difference between Sara's answer for z and Zoe's answer for s?

   Proposed by Serena An

   Solution: Sara gets $z = 6$, while Zoe solves $3s + 62 = 71$ for $s = 3$. The positive difference between z and s is $\boxed{3}$.

2. Toto has 10 boxes each containing 5 beans and 5 boxes each containing 10 beans. Each box contains one pinto bean. If Toto selects a box at random, and then selects a bean from that box at random, what is the probability that he selects a pinto bean?

   Proposed by Noah Walsh

   Solution: The probability that Toto selects a box containing 5 beans and then selects a pinto bean is $\frac{10}{15} \cdot \frac{1}{5} = \frac{2}{15}$. The probability that he selects a box containing 10 beans and then selects a pinto bean is $\frac{5}{15} \cdot \frac{1}{10} = \frac{1}{30}$. Summing, our answer is $\boxed{\frac{1}{6}}$.

3. In triangle GDL, I and E lie on sides GD, GL respectively such that $GI = 5$, $ID = 11$, $EG = 8$, $EL = 2$, and $IE = 7$. What is DL?

   Proposed by Serena An

   Solution: Since $GI \cdot GD = 5 \cdot 16 = 8 \cdot 10 = GE \cdot GL$, triangle GIE is similar to triangle GLD with ratio $\frac{1}{2}$, so $LD = 2 \cdot IE = \boxed{14}$.

4. Compute
$$\frac{1}{\frac{1}{41+9} + \frac{1}{41-9}} + \frac{1}{\frac{1}{41+40} + \frac{1}{41-40}}.$$

   Proposed by Ankit Bisain

   Solution: We have that
$$\frac{1}{\frac{1}{41+9} + \frac{1}{41-9}} + \frac{1}{\frac{1}{41+40} + \frac{1}{41-40}} = \frac{(41+9)(41-9)}{2 \cdot 41} + \frac{(41+40)(41-40)}{2 \cdot 41} = \frac{40^2 + 9^2}{2 \cdot 41} = \boxed{\frac{41}{2}}.$$

5. Mamamoo is a K-pop girl group consisting of the members Solar, Moonbyul, Wheein, and Hwasa. In their song "Wind Flower", the members sing one at a time for a total of 4 minutes of singing. Moonbyul sings for a third of amount of time that Wheein and Hwasa collectively sing. Given that Moonbyul sings for the least amount of time, what is the greatest possible difference between the number of seconds that Wheein and Hwasa sing?

   Proposed by Serena An

   Solution: Let s, m, w, h denote the fraction of the time that Solar, Moonbyul, Wheein, and Hwasa sing, respectively. Since $w, h \geq m = \frac{1}{3}(w + h)$, the greatest possible value of $|w - h|$ is $\frac{1}{3}(w + h)$, which occurs when Wheein gets $\frac{1}{3}(w + h)$ lines and Hwasa gets $\frac{2}{3}(w + h)$. Since $s \geq m$, we also have that
$$1 = s + m + w + h \geq \frac{1}{3}(w + h) + \frac{1}{3}(w + h) + (w + h),$$
so $w + h \leq \frac{3}{5}$. Then $\frac{1}{3}(w + h) \leq \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$. Finally, $\frac{1}{5}$ of a 4-minute song is $\boxed{48}$ seconds.

6. How many ways are there to arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8 in a circle such that no two adjacent numbers share a common factor greater than 1? Two arrangements are considered to be the same if they are rotations of each other.

   Proposed by Holden Mui

   Solution: No even numbers can be adjacent, and the 3 cannot be adjacent to the 6. The other three numbers can be placed in any fashion. Thus, this gives

   $$3! \times 2 \times 3! = \boxed{72}.$$

7. A class of students took a test, each receiving an integer grade from 0 to 10 inclusive. If the average score of the top 25 students was $9\frac{1}{5}$, and the average score of all the students was $6\frac{2}{7}$, what is the smallest possible number of students?

   Proposed by Max Lu

   Solution: The number of students N must be a multiple of 7, and we need $N \cdot 6\frac{2}{7} > 25 \cdot 9\frac{1}{5}$. We solve to get $N > \frac{805}{22} > 36$, so the minimal value of N is $\boxed{42}$. This is achievable, for example, by five students scoring a 10, twenty students scoring a 9, and seventeen students scoring a 2.

8. You and five of your friends line up in a random order to get tickets, one at a time. There are sixteen tickets available, labelled 1 through 16. Your friends do not care which ticket they get, and will take one at random; but your favorite number is 13, so if ticket #13 is available, you will take it. What is the probability that you get ticket #13?

   Proposed by Reagan Choi

   Solution: Note that you have a $\frac{1}{6}$ chance of lining up at each position. If there are n people ahead of you, they will take n of the 16 tickets at random, leading to an $\frac{n}{16}$ chance that ticket #13 is taken and a $\frac{16-n}{16}$ chance that you get ticket #13.

   Thus, since n ranges from 0 to 5, the probability that you get ticket #13 is

   $$\frac{1}{6}\left(\frac{16}{16}+\frac{15}{16}+\frac{14}{16}+\frac{13}{16}+\frac{12}{16}+\frac{11}{16}\right).$$

   This is the average of the values $\frac{16}{16}, \frac{15}{16}, \ldots, \frac{11}{16}$, which is $\boxed{\frac{27}{32}}$.

9. An acute triangle and a circle centered at the triangle's incenter both have an area of 4π. Given that $\frac{3}{4}$ of the circle's circumference lies outside the triangle, find the total area of the regions that are inside the triangle and outside the circle.

   Proposed by Robert Yang

   Solution: Because the sides of the triangle are equidistant from the incenter, each side cuts off one-fourth of the circumference, or a 90 degree arc. One 90 degree segment has area $\frac{4\pi}{4} - \frac{2\cdot 2}{2} = \pi - 2$. In addition, since the areas of the triangle and circle are equal, the desired area is also equal to the area of the regions inside the circle and outside the triangle which is $3(\pi - 2) = \boxed{3\pi - 6}$.

10. Let $ABCD$ be a rectangle, and let G be the centroid of ABC. Given that $DA = DG$, find $\frac{AD}{CD}$.

    Proposed by Maxim Li

    Solution: Let M be the midpoint of AC. Since M is also the midpoint of BD, and $BG = 2GM$, we have $AD = DG = \frac{2}{3} \cdot DB$. Squaring both sides, we get $AD^2 = \frac{4}{9}(AD^2 + CD^2)$, or $5AD^2 = 4CD^2$. Thus, $\frac{AD}{CD} = \sqrt{\frac{4}{5}} = \boxed{\frac{2\sqrt{5}}{5}}$.

11. Call a positive integer *redundant* if all of its digits appear at least twice in its base-10 representation. For example, 1102021 is redundant, but 11223 is not. Given that 2020 is the nth smallest redundant number, find n.

Proposed by Maxim Li

Solution: This is the same as the number of redundant numbers that are at most 2020. We proceed with casework on the number of distinct digits.

Case 1: There is 1 distinct digit.

Then there are 9 2-digit numbers $(11, 22, \ldots, 99)$, 9 3-digit numbers $(111, 222, \ldots, 999)$, and 1 4-digit number (1111), for a total of 19 redundant numbers.

Case 2: There are 2 distinct digits.

In this case, the number must have exactly 4 digits. If the number starts with 1, the redundant number is of the form $1aa1$, $1a1a$, or $11aa$ for a nonzero digit a, giving 27 redundant numbers. If the number starts with 2, then the only possibilities are 2002 and 2020. This gives a total of 29 redundant numbers.

If there are more than 2 distinct digits, the number has at least 6 digits and is greater than 2020. Thus, 2020 is the $19 + 29 = \boxed{48}$th redundant number.

12. A rectangular garden is built, and a rectangular walkway with fixed width is built around it. If the rectangles formed by the garden and walkway both have integral width, sides, and diagonals, what is the smallest possible area of the walkway?

    Proposed by Holden Mui

    Solution: It suffices to find two distinct integer right triangles with minimal difference in area such that the difference between their legs is the same. By exhausting all smaller cases, it can be checked the pair of integer right triangles $(5, 12, 13)$ and $(8, 15, 17)$ is minimal, so the answer is $8 \cdot 15 - 5 \cdot 12 = \boxed{60}$.

13. Define a sequence such that $a_1 = 2$, and for $n \geq 2$, a_n is the smallest possible positive integer greater than a_{n-1} such that there do not exist $1 \leq i, j \leq n$, not necessarily distinct, with $a_i + a_j = a_n$. Compute a_{2020}.

    Proposed by Rishabh Das

    Solution: We claim that the sequence consists of values that are either 2 or 3 $\pmod 5$.

    We prove this with induction. The base cases are $a_1 = 2$ and $a_2 = 3$, which hold.

    For the inductive hypothesis, assume that the only numbers in the sequence up to $5k + 3$ are those that are 2 or 3 $\pmod 5$. Now we may write

    $$\begin{aligned} 5k + 4 &= 2 + (5k + 2), \\ 5k + 5 &= 2 + (5k + 3), \text{ and} \\ 5k + 6 &= 3 + (5k + 3), \end{aligned}$$

    so the next term in the sequence must be at least $5k + 7$. Moreover, the sum of any two numbers in the sequence thus far is either $0, 1$, or $4 \pmod 5$, so $5k + 7$ and $5k + 8$ are then in the sequence, and the inductive step is complete.

    Then $a_{2020} = 5 \cdot 1010 - 2 = \boxed{5048}$.

14. Let x and y be real numbers such that $\cos x \neq 0$ and $\tan x = 3^{y+1}$. What is the largest possible value of $\sec x - 9^y$?

    Proposed by Luke Robitaille

    Solution: We have $9^y = \frac{(\tan x)^2}{9} = \frac{(\sec x)^2 - 1}{9}$, so by completing the square,

    $$\sec x - 9^y = \sec x - \frac{(\sec x)^2 - 1}{9} = \frac{85}{36} - \left(\frac{3}{2} - \frac{\sec x}{3}\right)^2 \leq \frac{85}{36}.$$

    We can check that with a suitable choice of x and y, we can achieve equality. Thus $\boxed{\frac{85}{36}}$ is the answer.

15. Serena tells Thomas and Ulysses that she is thinking of a two-digit positive integer. She truthfully and separately tells Thomas the tens digit, from which he can conclude that her number is not a perfect square, and Ulysses the units digit, from which he can also conclude that her number is not a perfect square. What is the sum of all possible numbers Serena is thinking of?

    Proposed by Alex Xu

    Solution: If the tens digit were 1, 2, 3, 4, 6, or 8, Thomas could not make such a conclusion, as the number could be 16, 25, 36, 49, 64, or 81; thus, the tens digit must be 5, 7, or 9. If the ones digit were 1, 4, 5, 6, or 9, Ulysses could not make such a conclusion, with the same counterexamples as above; thus, the ones digit must be 0, 2, 3, 7, or 8. This gives $3 \cdot 5 = 15$ total possibilities, with an average tens digit of $\frac{5+7+9}{3} = 7$ and an average ones digit of $\frac{0+2+3+7+8}{5} = 4$, for a final answer of $15 \cdot 74 = \boxed{1110}$.

16. Alice flips a fair coin until she flips her third "tails", at which point she stops. Given that she flips the coin at least five times and that the fifth flip was "heads," find the expected number of times that Alice flips the coin.

    Proposed by Reagan Choi

    Solution: The first four coin flips must contain either zero, one, or two tails (otherwise she would not flip the coin for a fifth time). These occur with probabilities $\frac{1}{16}$, $\frac{1}{4}$, and $\frac{3}{8}$, respectively.

    We will prove by induction that the expected number of flips it takes to flip k heads is $2k$. This is clearly true for $k = 0$. Now let $k \geq 1$ and let the expected value be E. Using the inductive hypothesis,

    $$E = \frac{1}{2}E + \frac{1}{2}(2k-2) + 1,$$

    so $E = 2k$.

    Now, the expected number of coin flips is

    $$5 + \frac{\frac{1}{16} \cdot 6 + \frac{1}{4} \cdot 4 + \frac{3}{8} \cdot 2}{\frac{1}{16} + \frac{1}{4} + \frac{3}{8}}.$$

    This evaluates to $\boxed{\frac{89}{11}}$.

17. There is a unique increasing sequence $p_1, p_2, \ldots p_k$ of primes for $k \geq 2$ such that $p_k = 13$ and

    $$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} - \frac{1}{p_1 p_2 \cdots p_k}$$

    is an integer. Compute $p_1 + p_2 + \cdots + p_k$.

    Proposed by Ankit Bisain

    Solution: Note that if we multiply by $p_1 \ldots p_k$, the expression becomes $\sum_{i=1}^{k} \frac{p_1 \ldots p_k}{p_i} - 1$, so by taking mod p_i, the product of the other $k-1$ primes must be 1 mod p_i. Thus, $13 | p_1 \ldots p_{k-1} - 1$. Now, mod 13, $2, 3, 5, 7, 11$ are $2^1, 2^4, 2^{-3}, 2^{-1}$ and 2^{-5}, respectively. Thus, the sets of other primes must be one of $\{2, 7\}, \{3, 5, 7\}, \{2, 3, 11\}$. Only the last one works, and thus the sum is $\boxed{29}$.

    Note: The problem initially did not have the condition that $k \geq 2$, so an answer of $\boxed{13}$ was accepted as well.

18. Let $ABCD$ be a convex quadrilateral with $AB = 6$, $BC = 9$, $CD = 15$, and $DA = 10$. Point P is chosen on line AC so that $PB = PD$. Given that $AC = 11$, find PB.

    Proposed by Maxim Li

    Solution: Let the tangent to (ABC) at B meet AC at T, and let the tangent to (ADC) meet AC at T'. Notice that $\frac{TA}{TC} = \left(\frac{TA}{TB}\right)\left(\frac{TB}{TC}\right) = \left(\frac{AB}{BC}\right)^2 = \frac{4}{9}$. Similarly, $\frac{T'A}{T'C} = \left(\frac{AD}{DC}\right)^2 = \frac{4}{9}$. Thus, $\frac{TA}{TC} = \frac{T'A}{T'C}$, so $T = T'$. But then $TB^2 = (TA)(TC) = TD^2$, so $TB = TD$. Thus, $T = P$. Now using $\frac{PA}{PC} = \frac{4}{9}$, we can calculate $PA = \frac{32}{5}$, and $PB = \frac{3}{2}PA = \boxed{\frac{48}{5}}$.

19. The polynomials

$$P(x) = x^3 - 7x^2 + ax + b$$

and

$$Q(x) = x^5 - 5x^4 + cx^3 + dx^2 + ex + f.$$

have the same set of roots, where a, b, c, d, e, f are real numbers. If $P(3) = 6$, find the sum of all possible values for $Q(3)$.

Proposed by Evan Chen and Max Lu

Solution: Let the roots of P be p, q, r. Then, the roots of Q either have two of p, q, r, each repeated once, or have one of p, q, r repeated twice.

We split into cases from here.

Case 1: The roots of Q are p, p, q, q, r. Then, by Vieta, since $p + q + r = 7$, and $2p + 2q + r = 5$, we have $r = 9$. So, $P(x) = (x - 9)(x^2 + kx + \ell)$, and $Q(x) = (x - 9)(x^2 + kx + \ell)^2$, for some k, ℓ. In this case, since $P(3) = 6$, we have $Q(3) = \frac{P(3)^2}{(3-9)} = -6$.

Case 2: The roots of Q are p, p, p, q, r. Again, by Vieta, $p + q + r = 7$, and $3p + q + r = 5$. So, $p = -1$. Thus, $P(x) = (x + 1)(x^2 + kx + \ell)$ and $Q(x) = (x + 1)^3(x^2 + kx + \ell)$ for some k, ℓ. In this case, $Q(3) = P(3)(3 + 1)^2 = 96$.

Thus, the sum of all possible values is $96 + (-6) = \boxed{90}$.

20. Let ABC be a triangle, and let the excenter opposite A be I_A. Points P, Q lie on the extensions of sides AB, AC past B, C respectively so that $BP = BC = CQ$. Suppose P, I_A, Q are collinear. Given that $AB = 20$ and $AC = 21$, find BC.

Proposed by Maxim Li

Solution: Let the incenter be I, and let BI, CI meet AC, AB at points D, E respectively. We can calculate that $AE = \frac{AB \cdot AC}{AC+BC}$ by the angle bisector theorem. Since $AQ = AC + BC$, we see that $AE \cdot AQ = AB \cdot AC$. But we also know $AI \cdot AI_A = AB \cdot AC$. Thus, AEI is similar to AI_AQ since $\angle EAI = \angle I_AAQ = \frac{1}{2}\angle A$. Similarly, we get ADI is similar to AI_AP. Thus, $\angle ADI + \angle AEI = \angle AI_AP + \angle AI_AQ = 180^\circ$, so A, D, I, E are concyclic. Since $\angle DIE = \angle BIC = 90^\circ + \frac{1}{2}\angle A$, this implies $90^\circ + \frac{3}{2}\angle A = 180^\circ$, or $\angle A = 60^\circ$. THen, by the law of cosines, $BC^2 = AB^2 + AC^2 - AB \cdot AC = 421$, so $BC = \boxed{\sqrt{421}}$.

Team Round

1. Let c be a real number. Suppose that the line with equation $y = cx$ is perpendicular to the line with equation $x + \frac{y}{8} = 2cx + 1$. What is the value of c?

   Proposed by Luke Robitaille

   Solution: We have that the lines $y = cx$ and $y = (16c - 8)x + 8$ are perpendicular, so their slopes multiply to -1. Thus $(16c - 8)c = -1$, so $(4c - 1)^2 = 0$ and $c = \boxed{\frac{1}{4}}$. (One can check that $c = \frac{1}{4}$ does indeed work.)

2. Two unit circles are given such that their common internal tangent has half the length of their common external tangent. Find the distance between their centers.

   Proposed by Holden Mui

   Solution: Let d be the desired distance. Then the given condition is

   $$d = 2\sqrt{d^2 - 2^2},$$

   so $d = \boxed{\frac{4\sqrt{3}}{3}}$.

3. Ethan has 5 different books: one yellow, two green, and two red. If the yellow book is between two books of the same color (but not necessarily adjacent to these two books), how many ways can he arrange the books on his shelf?

   Proposed by Isabella Quan

   Solution: We use complementary counting. There are $4! = 24$ ways for all non-yellow books to be on the left side of the yellow book. There are $2 \cdot 2 = 4$ ways for the two green books to be on the left of the yellow book and the two red books to be on the right of the yellow book. Summing and multiplying by 2 for symmetry gives 56 arrangements. Subtracting from the 120 total ways to arrange the five books gives our answer of $120 - 56 = \boxed{64}$.

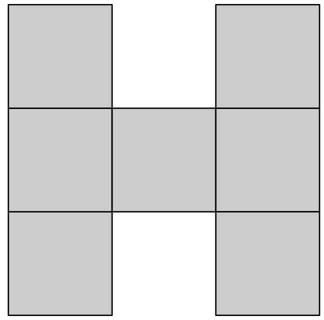
4. The cells of a 12×12 multiplication table are filled in such that the cell in the ith row and jth column is labeled with the product $i \cdot j$. How many of these products end with a 0?

   Proposed by Sean Li

   Solution: Note that a product ends with zero if and only if at least one of i, j is a multiple of 5, and at least one of i, j is even. Now, we have the cases where (i, j) is of the form $(5, j)$, $(i, 5)$, $(10, j)$, and $(i, 10)$. In the first two cases, j and i must be even, respectively, so there are 6 such cells in each case. In the third and fourth cases, i and j can be anything, so there are 12 cells in each of these cases.

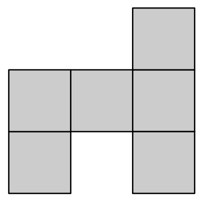
   However, each of the cells $(10, 10)$, $(10, 5)$, and $(5, 10)$ are counted, twice, so there are a total of $6 + 6 + 12 + 12 - 3 = \boxed{33}$ cells ending in 0.

5. Holden makes the letter H, a connected figure with seven blocks, as depicted. He then removes blocks one at a time without ever disconnecting the figure until all blocks are removed. Two blocks that only touch at a corner are not considered connected, so for example, Holden cannot remove the middle block or the blocks to its left or right on the first turn. How many different ways can he do this?

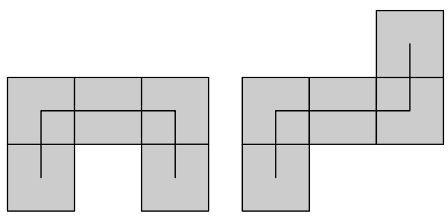


Proposed by Sean Li

Solution: The first move Holden makes must be to remove one of the four corners. Since they are symmetric, we will assume he removes the top-left block, and multiply our answer by 4 later. We are left with these blocks:



We must once again remove a corner. If we remove one of the two right corners, we will be left with a "string" of length 5.



("String" here means consecutive blocks such that a path can be drawn through them that doesn't hit any block more than once.) For each of these strings, there are 2^4 possibilities: on each of the next four steps, one of the two ends of the string must be removed, and on the last step only one block remains, which must be removed. Thus, these two cases in total give $2 \cdot 2^4 = 32$ possibilities.

Otherwise, the bottom-left corner block is removed. If one of the two right corners are removed from here, we obtain strings of length 4, which give 2^3 possibilities each, and so $2 \cdot 2^3 = 16$ in total. Otherwise, the left-center block is removed.

Now we must remove the center block, or one of the two right corner blocks. No matter which one we do, we are left with a string of length 3, which can be done in 2^2 ways, so this case gives $3 \cdot 2^2 = 12$ possibilities.

In total, there are $32 + 16 + 12 = 60$ different ways. Since we assumed which block was removed first, we multiply this by 4 to obtain $\boxed{240}$.

6. Let $ABCD$ be a rectangle with diagonals $AC = BD = 12$. Similar isosceles triangles EAB, FBC, GCD, HDA with bases AB, BC, CD, DA respectively are drawn outside the rectangle and have total area 24. If the areas of $ABCD$ and $EFGH$ are equal, find this common area.

   Proposed by Sanjana Das

   Solution: First, we let $AB = a$ and $BC = b$. Let the height of EAB be $\frac{1}{2}a \cdot x$, so the height of FBC is $\frac{1}{2}b \cdot x$. By symmetry, $EFGH$ is a rhombus, and its diagonals have length $a + bx$ and $b + ax$. By setting the areas of $ABCD$ and $EFGH$ equal, we have that $\frac{1}{2}(a+bx)(b+ax) = ab$. This simplifies to $abx^2 + (a^2+b^2)x = ab$; since $a^2+b^2 = 144$, we have $ab = 144 \cdot \frac{x}{1-x^2}$.

But EAB has area $\frac{1}{2} \cdot a \cdot \frac{1}{2}ax$ and BCF has area $\frac{1}{4}b^2x$, so the total area of the triangles is $\frac{1}{2}(a^2+b^2)x = 72x = 24$, which gives $x = \frac{1}{3}$. Finally, $ab = 144 \cdot \frac{3}{8} = \boxed{54}$.

7. A sequence of six prime numbers is given such that

   - the first term has two digits, and
   - each term is one more than twice its previous term.

   What is the last term?

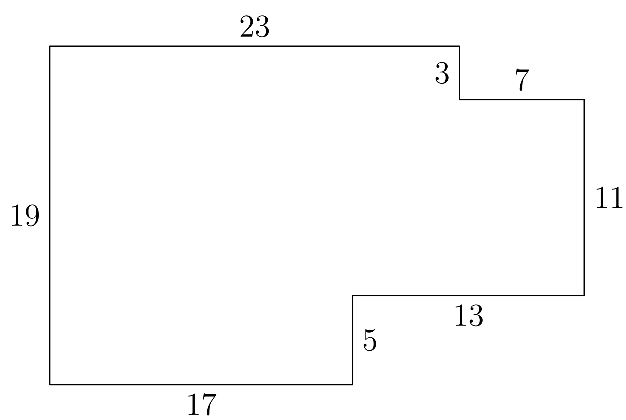
   Proposed by Holden Mui

   Solution: By computing the sequence mod 2, mod 3, and mod 5, it follows that the first term must be congruent to $-1 \pmod{30}$. Thus, the possible first terms are 29, 59, and 89. The sequence may not start with 29 or 59 because then the sequence would contain 119, which is composite. Thus, the first term must be 89, which makes the last term $\boxed{2879}$.

8. A polygon with all angles equal to 90° or 270° has the property that its sides have distinct prime lengths. Find the minimum possible perimeter.

   Proposed by Holden Mui

   Solution: Note that the polygon must have an even number of sides. It cannot be a rectangle, nor can it have six sides (or else 2 would be used as a length twice). The following rectangle uses the first odd eight primes as sides, so it must be minimal:



   Hence the answer is $3+5+7+11+13+17+19+23 = \boxed{98}$.

9. Let x, y, z be nonzero complex numbers such that $x+y+z = 5$, $x^2y + y^2z + z^2x = 20$, and $xy^2 + yz^2 + zx^2 = 19$. Compute
$$\frac{x^3+y^3+z^3-8}{xyz}.$$

   Proposed by Luke Robitaille

   Solution: We have

$$\begin{aligned} 125 &= (x+y+z)^3 \\ &= x^3+y^3+z^3+6xyz+3(x^2y+y^2z+z^2x)+3(xy^2+yz^2+zx^2) \\ &= x^3+y^3+z^3+6xyz+117, \end{aligned}$$

   so $x^3+y^3+z^3+6xyz = 8$, and $\frac{x^3+y^3+z^3-8}{xyz} = \boxed{-6}$.

10. Let P be a point on the incircle ω of unit equilateral triangle ABC. Let X be the second intersection of ω and $\overline{AP}$ and Y be the second intersection of ω and $\overline{BP}$. Find the maximum possible area of triangle PXY.

    Proposed by Holden Mui

    Solution: By symmetry, such a point can be found that makes PXY equilateral. Since PXY has fixed circumradius, such an equilateral triangle will have the maximal area. This equilateral triangle has side length $\frac{1}{2}$, so the answer is $\boxed{\frac{\sqrt{3}}{16}}$.

11. Let α be a real number satisfying $\alpha^3 = \alpha + 1$. Find $m^2 + n^2$ if $m < n$ are integers satisfying $\alpha^m + \alpha^n = 2$.

    Proposed by Holden Mui

    Solution: We have $\max(m, n) \leq 2$, because otherwise $a^m + a^n > 2$. Also $m \neq 0$, because otherwise we would have $m = n = 0$. If $m = 2$, then $n = -5$ is a solution since

    $$x^3 - x - 1 \mid x^2 - 2 + x^{-5}.$$

    Hence, $m = -5$, $n = 2$, and $m^2 + n^2 = \boxed{29}$.

12. Find the smallest positive integer with the property that the product of its prime divisors is equal to the sum of its nonprime proper divisors (a proper divisor is a positive integer divisor of a number, excluding itself).

    Proposed by Ankit Bisain

    Solution: The number must be squarefree, or else the product of its prime divisors would be one of the proper divisors. If $p_1, p_2, \ldots, p_k$ are the prime divisors, the problem statement is equivalent to

    $$\prod (p_i + 1) = \sum p_i + 2 \prod p_i.$$

    It's simple to check that $k = 1$ and $k = 2$ are impossible by Simon's Favorite Factoring Trick. For $k = 3$, by modulo 2, one of the primes is 2. If the number is $2pq$, SFFT gives $(p-2)(q-2) = 5$, giving

    $$2pq = 2 \cdot 3 \cdot 7 = 42.$$

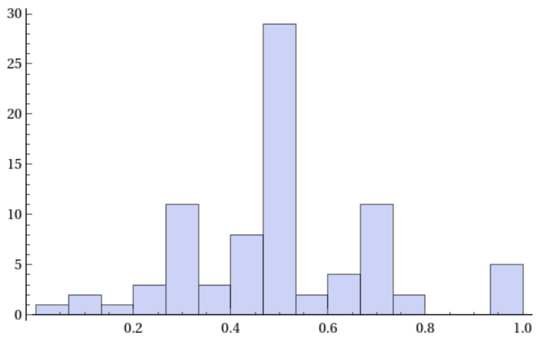
    Since since any number with 4 or more prime divisors has three prime divisors larger than $2, 3, 7$, $\boxed{42}$ is the minimal number.

13. Let X be the fraction of teams that will submit an answer less than $\frac{1}{2}$ to this problem. Estimate X. Your multiplier will be $0.9 + 0.2 \times \left(\min\left\{\frac{X}{Y}, \frac{Y}{X}\right\}\right)^2$, where Y is the answer you submit. If you submit nothing, an irrational number, or a number outside the range $(0, 1]$, your multiplier will default to 0.9.

    Solution: The correct value of X was $36/82 \approx 0.439$. Congratulations to the following teams for achieving a multiplier of ≥ 1.07:

    - Team **Knights Omni** from WWP High School North, who submitted $4/9 \approx 0.444$ and achieved a multiplier of 1.095.
    - Team **Barney** from Hunter College High School, who submitted $3/7 \approx 0.429$ and achieved a multiplier of 1.091.
    - Team **Low Expectations** from Hunter College High School, who submitted $5/11 \approx 0.455$ and achieved a multiplier of 1.087.
    - Team **Dream Team** from Stuyvesant High School, who submitted $7/15 \approx 0.467$ and achieved a multiplier of 1.077.
    - Team **card duck curls penguin** from Lynbrook High School, who submitted 0.4075 and achieved a multiplier of 1.072.

    Shown below is a histogram of the data set.



$N = 82$
Mean $= 0.503$
StdDev $= 0.198$
$Q_1 = 0.38$
Median $= 0.5$
$Q_3 = 0.6$

Some fun facts:

- Of the 111 teams that submitted, 29 teams either submitted nothing or an answer not in the interval $(0, 1]$, thereby earning a multiplier of 0.9.
- The three most common submissions were $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, submitted 16, 7, 5 times respectively.

*NEMO 2020 was directed and run by Serena An, Rishabh Das, Sean Li, Max Lu, Holden Mui, and Eric Shen. We would like to thank Alex Xu, Ankit Bisain, Derek Liu, Eric Yang, Isabella Quan, Luke Robitaille, Maxim Li, Maxwell Sun, Noah Walsh, Raymond Feng, Reagan Choi, Robert Yang, and Sanjana Das for submitting problem proposals. We would also like to thank Alex Yi, Alex Zhao, Brian Liu, Daniel Xu, Evan Chen, Jaedon Whyte, Kevin Min, and William Wang for helping with problem selection and testsolving. Lastly, we would like to congratulate the **437** participants and **111** teams from **47** different schools for participating in this edition of NEMO.*