Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 20 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 20 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. Find the sum of the fourth smallest prime number and the fourth smallest composite number.

2. A daycare has babies that are one year, two years, and four years old. If the two-year-old babies are excluded, the average age of the remaining babies is 2.0; if the four-year-old babies are excluded, the average age of the remaining babies is 1.9; and if the one-year-old babies are excluded, the average age of the remaining babies is A. Compute A.

3. Andrew encounters a square $ABCD$, with side length one meter, on a sidewalk. He marks a point E such that B is on $\overline{DE}$ and $EB = \frac{1}{2}BD$. He then draws segments DE, EA, and AC with chalk, thus forming a giant four. Compute the total length, in meters, that Andrew has drawn.

4. Real numbers x, y, z satisfy the inequalities

$$-8 < x < 5, \quad -2 < y < 3, \quad -5 < z < 6.$$

There exist real numbers m and n such that $m < x \cdot y \cdot z < n$ for all choices of x, y, z. Find the minimum possible value of $n - m$.

5. Find the fifth smallest positive integer N such that the sum of the digits of N^2 is equal to the sum of the digits of N.

6. If $\dfrac{x}{y} = \dfrac{3}{x} = \dfrac{y}{4}$, compute the value of x^3.

7. Carol has a fair die with faces labeled 1 to 6. She rolls the die once, and if she rolls a 1, 2, or 3, she rolls the die a second time. What is the expected value of her last roll?

8. A *polyomino* is a geometric figure formed by joining one or more unit squares edge to edge. A *frame* is a polyomino with one hole such that the both its outside boundary and inside boundary are rectangles. Find the largest integer n for which there does not exist a frame with area n.

9. Let $ABCD$ be a square with center O, and let P be a randomly chosen point in the square's interior. What is the probability that triangles AOP, BOP, COP, and DOP are all obtuse?

10. Team MOP and Team MOSP are the last teams standing in a shouting tournament. The championship match consists of at most five rounds and will end when a team wins three rounds. Given that both teams have an equal chance of winning each round and that Team MOP won the fourth round, what is the probability that Team MOP won the tournament?

11. If n is a positive integer with d digits, let $p(n)$ denote the number of distinct d-digit numbers whose digits are a permutation of the digits of n. Find the smallest value of N such that $p(N) \geq 2019$.

12. Vincent chooses two (not necessarily distinct) positive divisors of 360 independently and at random. What is the probability that they are relatively prime?

13. If a and b are real numbers such that

$$\frac{a}{b} + \frac{b}{a} = 20 \quad \text{and} \quad \frac{a^2}{b} + \frac{b^2}{a} = 19,$$

then compute $|a - b|$.

14. Find all primes $p \geq 5$ such that p divides $(p-3)^{p-3} - (p-4)^{p-4}$.

15. The county of NEMO-landia has five towns, with no roads built between any two of them. How many ways can the NEMO-landian board build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

16. Externally tangent circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have radii 6 and 8, respectively. A line ℓ intersects $\mathcal{C}_1$ at A and B, and $\mathcal{C}_2$ at C and D. Given that $AB = BC = CD$, find AD.

17. Suppose x and y are positive reals satisfying

$$\sqrt{xy} = x - y = \frac{1}{x+y} = k.$$

Determine k.

18. Paper triangle ABC, with $\angle A > \angle B > \angle C$, has area 90. If side AB is folded in half, the area of the resulting figure is 54. If side BC is folded in half, the area of the resulting figure is 60. What is the area of the resulting figure if side CA is folded in half?

19. Find all reals $x > 1$ such that $\log_2\left(\log_2 x^8\right) \cdot \log_x\left(x^4 \log_x 2\right) = 12$.

20. Let ABC be an equilateral triangle and P a point in its interior obeying $AP = 5$, $BP = 7$, and $CP = 8$. Line CP intersects $\overline{AB}$ at Q. Compute AQ.

Team Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 12 problems that are to be completed in 30 minutes, along with a small minigame that will generate a multiplier for a team's team round score.
- To ensure that your answers to problems 1 to 12 are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- The minigame, or problem 13, will ask your team to submit a pair of real numbers (x, y), and will generate a multiplier between 0.9 and 1.1 for your team round score. Be sure to try out the minigame, as submitting nothing will decrease your team round score.
- Each of the 12 team round problems have a predetermined point value; your team's team round score will be the sum of the point values assigned to each question that is correctly answered, multiplied by the multiplier generated by the minigame. Excluding the multiplier, a perfect score (achieved through answering all 12 problems correctly) is **400 points**.
- Your team score will be a combination of your score on the team round and the scores of each individual member.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your team name, team members' names, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. **[22]** How many ways can Danielle select two pets of different color from five brown dogs, seven grey kittens, and eight yellow parakeets? Two animals of the same color are still considered distinguishable.

2. **[24]** Find the unique three-digit positive integer which
   - has a tens digit of 9, and
   - has three distinct prime factors, one of which is also a three-digit positive integer.

3. **[26]** A bin of one hundred marbles contains two golden marbles. Shen, Li, and Park take turns, in that order, removing marbles from the bin. If a player draws a golden marble, the game ends and the player wins. What is the probability that Shen wins?

4. **[28]** Points $A(5,1)$, $B(1,7)$, $C(3,10)$, and $D(7,10)$ are on the coordinate plane. Circle γ is tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. What is the area of γ?

5. **[30]** Jerry picks positive integers a, b, c. It turns out that $\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$ and the cubic $x^3 - ax^2 + bx - c$ has distinct integer roots. What is the smallest possible value of $a + b + c$?

6. **[32]** Compute $\sqrt{73^3 - 2^4 \cdot 3^7}$, given that it is an integer.

7. **[34]** Nine fair coins are flipped, and the coins are randomly divided into three piles of three coins. What is the expected number of heads in the pile with the largest number of heads?

8. **[36]** What is the sixth smallest positive real x such that $x \cdot \lfloor x \rfloor \cdot \{x\} = 6$? (Here, $\lfloor x \rfloor$ and $\{x\}$ denote the integer and fractional parts of x, respectively.)

9. **[38]** Triangle ABC has $AB = 6$, $BC = 11$, and $CA = 7$. Let M be the midpoint of $\overline{BC}$. Points E and O are on $\overline{AC}$ and $\overline{AB}$, respectively, and point N lies on line AM. Given that quadrilateral $NEMO$ is a rectangle, find its area.

10. **[40]** Daniel paints each of the nine triangles in the diagram either crimson, scarlet, or maroon. Given that any pair of triangles sharing a side are painted different colors, how many ways can Daniel paint the diagram? Two diagrams that differ by a rotation or reflection are considered distinct.

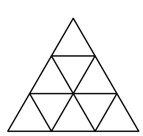
11. **[42]** An S-tetromino is inscribed in square $ABCD$ such that their perimeters share exactly four points, one of which is X. Given that A, B, and X are collinear, maximize $\frac{AX}{BX}$. (An *S-tetromino* is a geometric figure comprised of four unit squares, joined edge to edge in the pictured fashion.)

12. **[44]** Let n be a positive integer. Points A, B, C are selected from the unit n-dimensional hypercube

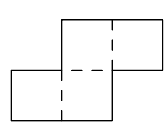
$$\mathcal{H} = \{(x_1, x_2, \ldots, x_n) \mid 0 \le x_i \le 1 \text{ for } i = 1, \ldots, n\}.$$

Given that the maximum possible area of $\triangle ABC$ is a positive integer, minimize n.

13. Pick a point $P = (x, y)$ such that $0 \le x, y \le 1$. Let S be the set of points submitted by all teams, $f(A)$ be the minimum distance from A to any other point in S, and let D be the maximum value of $f(A)$ across all points A in S. Your multiplier will be $0.9 + 0.01 \times \left\lfloor \frac{20 f(P)}{D} \right\rfloor$. If you submit nothing or pick an invalid point, your multiplier will default to 0.9. In particular, you would like your point to be as far away from the closest other submitted point as possible.



Problem 10.



Problem 11.

Answers

Individual Round

1. 16
2. 40/19
3. $\dfrac{5\sqrt{2}+\sqrt{10}}{2}$
4. 264
5. 19
6. 36
7. 17/4
8. 9
9. $\dfrac{\pi-2}{4}$
10. 3/4
11. 1012345
12. 35/192
13. $\dfrac{3\sqrt{11}}{11}$
14. 73
15. 222
16. $\sqrt{429}$
17. $\dfrac{\sqrt[4]{125}}{5}$
18. 63
19. $2, \sqrt{2}, \sqrt[4]{2}$
20. $\dfrac{5\sqrt{129}}{13}$

Team Round

1. 131
2. 894
3. 17/50
4. 9π
5. 63
6. 595
7. 141/64
8. $\dfrac{8+\sqrt{67}}{2}$
9. $\dfrac{77\sqrt{10}}{27}$
10. 528
11. 8
12. 7

Solutions

Individual Round

1. Find the sum of the fourth smallest prime number and the fourth smallest composite number.

   Proposed by Brandon Wang

   Solution: The first four primes are $2, 3, 5, 7$, while the first four composite numbers are $4, 6, 8, 9$. The answer is thus $7 + 9 = \boxed{16}$.

2. A daycare has babies that are one year, two years, and four years old. If the two-year-old babies are excluded, the average age of the remaining babies is 2.0; if the four-year-old babies are excluded, the average age of the remaining babies is 1.9; and if the one-year-old babies are excluded, the average age of the remaining babies is A. Compute A.

   Proposed by Sean Li

   Solution: Let there be x one-year-olds, y two-year-olds, and z four-year-olds. Then we have

   $$\frac{x+4z}{x+z} = 2 \implies x = 2z \quad \text{and} \quad \frac{x+2y}{x+y} = 1.9 \implies 9x = y.$$

   Thus, we have that $A = \frac{2y+4z}{y+z} = \frac{18x+2x}{9x+\frac{1}{2}x} = \boxed{\frac{40}{19}}$.

3. Andrew encounters a square $ABCD$, with side length one meter, on a sidewalk. He marks a point E such that B is on $\overline{DE}$ and $EB = \frac{1}{2}BD$. He then draws segments DE, EA, and AC with chalk, thus forming a giant four. Compute the total length, in meters, that Andrew has drawn.

   Proposed by Brandon Wang

   Solution: We assign the coordinates $A(0, 1)$, $B(1, 1)$, $C(1, 0)$, and $D(0, 0)$. Then $E = \frac{3}{2}B - \frac{1}{2}D = (\frac{3}{2}, \frac{3}{2})$. It is then straightforward to compute (with the Distance Formula) that $DE = \frac{3}{2}\sqrt{2}$, $EA = \frac{1}{2}\sqrt{10}$, and $AC = \sqrt{2}$, for a total of $\boxed{\frac{5\sqrt{2}+\sqrt{10}}{2}}$.

4. Real numbers x, y, z satisfy the inequalities

   $$-8 < x < 5, \quad -2 < y < 3, \quad -5 < z < 6.$$

   There exist real numbers m and n such that $m < x \cdot y \cdot z < n$ for all choices of x, y, z. Find the minimum possible value of $n - m$.

   Proposed by Sean Li

   Solution: We multiply the extrema of x, y, z to get various local maxima and minima of xyz; we can then find the global maximum and minimum. (More rigorously, these are infinima and suprema, but the semantics don't matter as much.)

   The global maximum consists of the product of two negative extrema and a positive extrema, or all three positive. These return $(5)(-2)(5) = 100$, $(-8)(3)(-5) = 120$, $(-8)(-2)(6) = 96$, and $(5)(3)(6) = 90$. The maximum is thus 120, achieved as $(x, y, z) \to (-8, 3, -5)$. Similarly, the global minimum is -144, at $(x, y, z) \to (-8, 3, 6)$. So $(m, n) = (-144, 120)$ and $n - m = \boxed{264}$.

5. Find the fifth smallest positive integer N such that the sum of the digits of N^2 is equal to the sum of the digits of N.

   Proposed by Eric Gan

   Solution: The sum of the digits of a positive integer n is congruent to $n \pmod 9$. Thus, we have that $N^2 \equiv N \pmod 9$, or that $N \equiv 0, 1 \pmod 9$. The first five such numbers are $1, 9, 10, 18, 19$, all of which work. Thus, the answer is $\boxed{19}$.

6. If $\dfrac{x}{y} = \dfrac{3}{x} = \dfrac{y}{4}$, compute the value of x^3.

   Proposed by Serena An

   Solution: Note that $\left(\frac{3}{x}\right)^2 = \frac{y}{4} \cdot \frac{x}{y}$, id est $\frac{9}{x^2} = \frac{x}{4}$. Thus, $x^3 = \boxed{36}$.

7. Carol has a fair die with faces labeled 1 to 6. She rolls the die once, and if she rolls a 1, 2, or 3, she rolls the die a second time. What is the expected value of her last roll?

   Proposed by Sean Li

   Solution: There is a $(1/2)^2$ chance that her final roll is one of $\{1, 2, 3\}$, and thus a $3/4$ chance that her final roll is one of $\{4, 5, 6\}$. Then by the definition of expected value, our answer is $\frac{1}{4}\left(\frac{1+2+3}{3}\right) + \frac{3}{4}\left(\frac{4+5+6}{3}\right) = \boxed{\frac{17}{4}}$.

8. A *polyomino* is a geometric figure formed by joining one or more unit squares edge to edge. A *frame* is a polyomino with one hole such that the both its outside boundary and inside boundary are rectangles. Find the largest integer n for which there does not exist a frame with area n.

   Proposed by Holden Mui

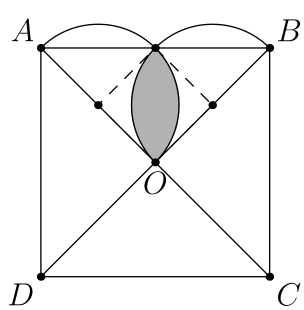
   Solution: The answer is $n = \boxed{9}$. Say that its outside boundary is an $a \times b$ rectangle with $a < b$. Then its inner boundary is at most $(a-2) \times (b-2)$, so the frame has area $\geq 2a + 2b - 4$, which is always even. The second smallest frame is achieved with an inner boundary of $(a-2) \times (b-3)$, which yields $3a + 2b - 6$. It is then clear that $n = 9$ is not possible: we must have $3a + 2b - 6 = 9$, which is not possible for $a > 2, b > 3$. The constructions for $n > 9$ are

   - $3 \times \frac{n-2}{2} - 1 \times \frac{n-6}{2}$ for even n, and
   - $3 \times \frac{n-3}{2} - 1 \times \frac{n-9}{2}$ for odd n.

9. Let $ABCD$ be a square with center O, and let P be a randomly chosen point in the square's interior. What is the probability that triangles AOP, BOP, COP, and DOP are all obtuse?

   Proposed by Sean Li

   Solution: Without loss of generality $OA = OB = OC = OD = 2$ and P is in $\triangle AOB$; then $\triangle COP$ and $\triangle DOP$ are always obtuse. Triangle AOP is obtuse if and only if P lies within the semicircle with diameter $\overline{AO}$, and similarly for triangle BOP. So P must lie within the shaded region, which has area $\frac{1}{2}\pi - 1$. This is over a region of area 2, for a total probability of $\boxed{\frac{\pi-2}{4}}$.



10. Team MOP and Team MOSP are the last teams standing in a shouting tournament. The championship match consists of at most five rounds and will end when a team wins three rounds. Given that both teams have an equal chance of winning each round and that Team MOP won the fourth round, what is the probability that Team MOP won the tournament?

    Proposed by Serena An

Solution: We perform casework as to how the tournament could have looked, given that Team MOP won the fourth round. Let M and S be shorthand for Team MOP and MOSP, respectively.

- If Team MOSP wins, then they must have won a fifth round (i.e. of the form $\_\ \_\ \_\ M\ S$) and must have won exactly two of the first three rounds; if they won the first three rounds, then the tournament would not have a fourth round. This happens with $\frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$.
- If Team MOP wins, then there are two further cases:
  - There are only four rounds in the tournament. Then Team MOP must have won exactly two of the first three rounds; this happens with probability $\frac{3}{8}$.
  - There are five rounds in the tournament. Then Team MOP must have won exactly one of the first three rounds, and the last round; this happens with probability $\frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$.

In summary, the answer is $\frac{3/8+3/16}{3/16+3/8+3/16} = \boxed{\frac{3}{4}}$.

11. If n is a positive integer with d digits, let $p(n)$ denote the number of distinct d-digit numbers whose digits are a permutation of the digits of n. Find the smallest value of N such that $p(N) \geq 2019$.

    Proposed by Rishabh Das

    Solution: The answer is $\boxed{1012345}$, which yields $p(N) = 2160 \geq 2019$. We now show it is the least possible N.

    Note that N must have at least seven digits, as $p(m) \leq 6! = 720$ for all $m < 10^6$. Moreover, any seven-digit m with at least two zeroes in its decimal representation has $p(m) \leq \binom{6}{2} \cdot 5! = 1800 < 2019$ by considering where the zeroes go.

    Thus, N is more than 1010000 and has at most one zero in its decimal representation. If $1010000 \leq N < 1020000$, then N has at least two ones and a zero in its representation, so it has at most $\binom{6}{1} \cdot \binom{6}{2} \cdot 4! = 2160$ permutations, with equality when there are exactly two ones, one zero, and all other digits nonzero and distinct. If we have any more duplicates, then this number is reduced by a factor of at least $\frac{1}{3}$, so we must have equality. The least number satisfying these conditions is 1012345, as desired.

12. Vincent chooses two (not necessarily distinct) positive divisors of 360 independently and at random. What is the probability that they are relatively prime?

    Proposed by Ankit Bisain

    Solution: Factorize $360 = 2^3 3^2 5^1$. Let the two numbers that Vincent chooses be $M = 2^a 3^b 5^c$ and $N = 2^x 3^y 5^z$. Then we must have $\gcd(M, N) = 1$, or

    $$\min(a, x) = \min(b, y) = \min(c, z) = 0.$$

    There are $2 \cdot (3+1) - 1 = 7$ pairs $0 \leq a, x \leq 3$ that satisfy $\min(a, x) = 0$. Similarly, there are 5 working pairs (b, y) and 3 working pairs (c, z), for a total of $7 \cdot 5 \cdot 3 = 105$ relatively prime pairs (M, N). This is over $((3+1)(2+1)(1+1))^2 = 576$ pairs of divisors, for a total probability of $\frac{105}{576} = \boxed{\frac{35}{192}}$.

13. If a and b are real numbers such that

    $$\frac{a}{b} + \frac{b}{a} = 20 \quad \text{and} \quad \frac{a^2}{b} + \frac{b^2}{a} = 19,$$

    then compute $|a - b|$.

    Proposed by Rishabh Das

Solution: We multiply the first equation by $a+b$, yielding

$$20(a+b) = \left(\frac{a}{b}+\frac{b}{a}\right)(a+b) = \frac{a^2}{b}+\frac{b^2}{a}+(a+b) = 19+(a+b).$$

So $a+b=1$. Adding 2 to the first equation, we get $\frac{a^2+2ab+b^2}{ab} = 22$, id est $ab = \frac{1}{22}$. Thus, $(a-b)^2 = (a+b)^2 - 4ab = 1 - \frac{4}{22} = \frac{9}{11}$, so $|a-b| = \boxed{\frac{3\sqrt{11}}{11}}$, as desired.

14. Find all primes $p \geq 5$ such that p divides $(p-3)^{p-3} - (p-4)^{p-4}$.

Proposed by Sean Li

Solution: Observe that for $p \neq 2, 3$,

$$(p-3)^{p-3} - (p-4)^{p-4} \equiv (-3)^{p-3} - (-4)^{p-4} \overset{\text{FLT}}{\equiv} (-3)^{-2} - (-4)^{-3} \equiv \frac{1}{9} + \frac{1}{64} \equiv \frac{73}{576} \pmod{p}$$

which quickly yields $p = \boxed{73}$ as the only solution.

15. The county of NEMO-landia has five towns, with no roads built between any two of them. How many ways can the NEMO-landian board build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

Proposed by Sean Li

Solution: We instead count the complement. The key claim is that the county board fails if and only if there is some town with no road connected to it. Indeed, say this were not the case. Then the county is decomposable into k connected parts with $c_1, \ldots, c_k$ towns (all greater than 1). Then we have at most $\sum_{i=1}^{k} \binom{c_i}{2} \leq \binom{2}{2} + \binom{3}{2} < 5$ roads, contradiction.

Thus, we can choose a town to have no roads, then pick one of the $\binom{4}{2} = 6$ possible roads between the remaining four cities not to build. Then the county board can fail in $5 \cdot 6 = 30$ ways, so it can succeed in $\binom{10}{5} - 30 = \boxed{222}$ ways.

16. Externally tangent circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have radii 6 and 8, respectively. A line ℓ intersects $\mathcal{C}_1$ at A and B, and $\mathcal{C}_2$ at C and D. Given that $AB = BC = CD$, find AD.

Proposed by Sean Li

Solution: Let $AB = BC = CD = 2x$, so we wish to find $6x$. The crux move is to drop altitudes from X and Y onto ℓ; say their feet are at S and T, respectively. Then evidently $BS = CT = x$, id est $ST = 4x$. Moreover, via the Pythagorean Theorem we calculate $SX = \sqrt{36-x^2}$ and $TY = \sqrt{64-x^2}$. Let the foot of X onto $\overline{TY}$ be Z. Because $STZX$ is a rectangle, we get $ST = XZ$, $ZY = TY - SX$. Therefore,

$$XZ^2 + ZY^2 = XY^2 \implies (4x)^2 + \left(\sqrt{64-x^2} - \sqrt{36-x^2}\right)^2 = 14^2,$$

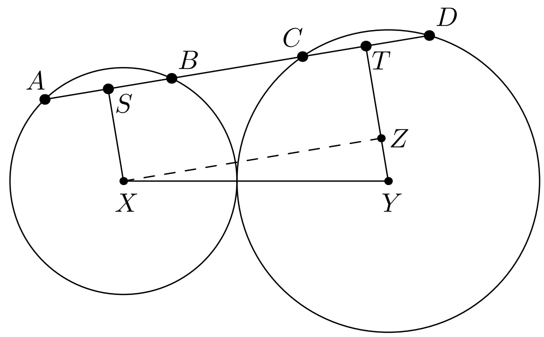
or $x = \pm\sqrt{429}/6$. Taking the positive solution, we get $6x = \boxed{\sqrt{429}}$, as desired.

17. Suppose x and y are positive reals satisfying

$$\sqrt{xy} = x - y = \frac{1}{x+y} = k.$$

Determine k.

Proposed by Holden Mui



Solution: Firstly, note that

$$\begin{aligned}(x+y)^4 &= \left((x-y)^2+4xy\right)(x+y)^2 \\ &= \left(\left(\frac{1}{x+y}\right)^2+4\left(\frac{1}{x+y}\right)^2\right)(x+y)^2 \\ &= 5.\end{aligned}$$

So $k = \frac{1}{x+y} = \frac{1}{\sqrt[4]{5}} = \boxed{\frac{\sqrt[4]{125}}{5}}$.

18. Paper triangle ABC, with $\angle A > \angle B > \angle C$, has area 90. If side AB is folded in half, the area of the resulting figure is 54. If side BC is folded in half, the area of the resulting figure is 60. What is the area of the resulting figure if side CA is folded in half?

    Proposed by Ishika Shah

    Solution: Let D, E, F be the feet of the altitudes from A, B, C, respectively.

    First consider when the triangle is folded across $\overline{AB}$; the triangle is folded across ℓ, the perpendicular bisector of $\overline{AB}$. Because $\angle A > \angle B$, A and C are on the same side of ℓ. As such, let ℓ meet $\overline{AB}$ and $\overline{BC}$ at M_C and N_C. Thus

$$1 - \frac{54}{90} = \frac{[M_CN_CB]}{[ABC]} = \frac{\frac{1}{2}M_CN_C \cdot BM_C}{\frac{1}{2}CF \cdot AB} = \frac{M_CN_C}{CF} \cdot \frac{BM_C}{AB} = \frac{1}{2}\frac{BM_C}{BF}.$$

    So $BF = \frac{5}{4}BM_C = \frac{5}{8}AB$, or $AF : BF = 3 : 5$. Similarly, $BF : CF = 1 : 3$, so $CF : AF = 1 : 5$ by Ceva's. Finally, the area of $\triangle M_BC_BC$ is $\frac{1}{2} \cdot \frac{1+\frac{1}{5}}{2} \cdot 90 = 27$, so the answer is $90 - 27 = \boxed{63}$.

19. Find all reals $x > 1$ such that $\log_2\left(\log_2 x^8\right) \cdot \log_x\left(x^4 \log_x 2\right) = 12$.

    Proposed by Sean Li

    Solution: Rewrite the equation as $(\log_2(\log_2 x) + 3)(\log_x(\log_x 2) + 4) = 12$. Substitute $x = 2^{2^t}$ to get

$$(t+3)\left(-\frac{t}{2^t}+4\right) = 12 \implies -\frac{t^2}{2^t} + 4t - \frac{3t}{2^t} = 0,$$

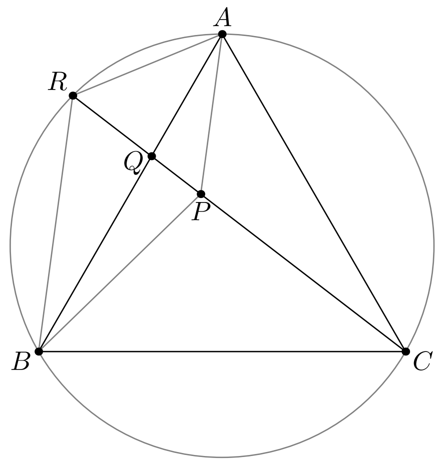
    which yields $t(2^{t+2} - t - 3) = 0$, so either $t = 0$ or $2^{t+2} = t + 3$. Observe that the latter equation works for $t = -1, -2$; these are unique as 2^{t+2} is convex. Thus, $x = 2^{2^0}, 2^{2^{-1}}, 2^{2^{-2}} = \boxed{2, \sqrt{2}, \sqrt[4]{2}}$ all work.

20. Let ABC be an equilateral triangle and P a point in its interior obeying $AP = 5$, $BP = 7$, and $CP = 8$. Line CP intersects $\overline{AB}$ at Q. Compute AQ.

    Proposed by Eric Shen

Solution: Let R be the image of P under $60°$ clockwise rotation at A (assuming $\angle CAB$ is counter-clockwise). Notice that since $AP = AR = 5$ and $\angle PAR = 60°$, $\triangle APR$ is an equilateral triangle. Specifically, $PR = 5$. Now $BR = 8$ and $BP = 7$, so it is well-known that $\angle BRP = 60°$. An easy way to see this is just to apply the Law of Cosines on $\triangle BPR$.

Note that $\angle APR = 60°$ and $\angle APC = 120°$, so C, P, R collinear. Thus $\overline{RC}$ bisects $\angle ARB$, thus $AQ/BQ = AR/BR = 5/8$. Finally by the Law of Cosines on $\triangle ARB$, $AB^2 = 5^2 + 8^2 + 5 \cdot 8 = 129$, so $AQ = \boxed{\frac{5\sqrt{129}}{13}}$, as desired.



Team Round

1. How many ways can Danielle select two pets of different color from five brown dogs, seven grey kittens, and eight yellow parakeets? Two animals of the same color are still considered distinguishable.

   Proposed by Sean Li

   Solution: We can choose a dog and kitten, a kitten and parakeet, or a parakeet and dog. By the Multiplication Rule, the total number of ways to achieve one of the three distinct possibilities is $5 \cdot 7 + 7 \cdot 8 + 8 \cdot 5 = \boxed{131}$.

2. Find the unique three-digit positive integer which

   - has a tens digit of 9, and
   - has three distinct prime factors, one of which is also a three-digit positive integer.

   Proposed by Sean Li

   Solution: Let the three-digit number in question be N, and let its largest prime factor be p. Then its other two prime factors multiply to $N/p < 999/101 < 10$. However, the only single-digit number with two prime factors is 6, so $N = 6p > 600$.

   We then perform cases on what the hundreds digit of N is.

   - If N has hundreds digit 6, then because $6 \mid N$ we must have $N \in \{690, 696\}$, neither of which yield prime $N/6$.
   - If N has hundreds digit 7, then we must have $N \in \{792, 798\}$, neither of which yield prime $N/6$.
   - If N has hundreds digit 8, then we must have $N = 894$, which yields $N/6 = 149$, a prime.
   - If N has hundreds digit 9, then we must have $N \in \{990.996\}$, neither of which yield prime $N/6$.

   Thus, our answer is $\boxed{894}$.

3. A bin of one hundred marbles contains two golden marbles. Shen, Li, and Park take turns, in that order, removing marbles from the bin. If a player draws a golden marble, the game ends and the player wins. What is the probability that Shen wins?

   Proposed by Sean Li

   Solution: We instead consider the equivalent game where players remove the first remaining marble in a line of one hundred marbles. Then Shen wins if and only if the first golden marble is in the $(3n + 1)$-th position for some integer n.

   Moreover, if the first golden marble is in position $3n + 1$, then the second marble can be anywhere from the $(3n + 2)$-th position to the 100-th position, for a total of $99 - 3n$ positions. Thus, there are

   $$\sum_{n=0}^{33}(99 - 3n) = 3 \cdot \sum_{n=0}^{33} n = \frac{3 \cdot 33 \cdot 34}{2}$$

   satisfactory lines of marbles out of $\binom{100}{2}$, for a probability of $\frac{\frac{3 \cdot 33 \cdot 34}{2}}{\binom{100}{2}} = \boxed{\frac{17}{50}}$.

   Remark. Coincidentally, the probability that Shen wins with one golden marble and 99 other marbles is also 17/50.

4. Points $A(5, 1)$, $B(1, 7)$, $C(3, 10)$, and $D(7, 10)$ are on the coordinate plane. Circle γ is tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. What is the area of γ?

   Proposed by Sean Li

   Solution: Note that the center of Γ lies on the angle bisector of $\angle ABC$, which is the line $y = 7$. The distance from a point on the line and $\overline{CD}$ is always 3, so the area of γ must be $\pi(3)^2 = \boxed{9\pi}$.

5. Jerry picks positive integers a, b, c. It turns out that $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$ and the cubic $x^3 - ax^2 + bx - c$ has distinct integer roots. What is the smallest possible value of $a + b + c$?

   Proposed by Rishabh Das

   Solution: Let the roots be r_1, r_2, r_3. By Vieta's formulas, $r_1 + r_2 + r_3 = a, r_1r_2 + r_2r_3 + r_3r_1 = b, r_1r_2r_3 = c$. Also let $f(x) = x^3 - ax^2 + bx - c$.

   Note that if any of the r_i are even, then c is even. Also note that one of a or b will also be even, a contradiction. Thus, all roots of f are odd. Moreover, the cubic cannot have negative roots, since $f(-k)$ will be a negative number for positive k, so all roots of f are positive.

   We look at the smallest positive integers that can be written as the product of three distinct odd positive integers. The smallest one of these is 15, where $r_1 = 1, r_2 = 3, r_3 = 5$. This means $a = 9$ and $c = 15$, a contradiction.

   The next smallest integer with this property is 21, where $r_1 = 1, r_2 = 3, r_3 = 7$. We see $a = 11, b = 31, c = 21$, and this does work, yielding $a + b + c = \boxed{63}$. The next smallest integer with this property is 27, and it is not hard to see that for $c \geq 27$ we must have $a + b + c \geq 63$.

6. Compute $\sqrt{73^3 - 2^4 \cdot 3^7}$, given that it is an integer.

   Proposed by Ankit Bisain

   Solution: Here is a solution with almost no computation.

   Let $n = \sqrt{73^3 - 2^4 \cdot 3^7}$ and consider n modulo 5, 7, and 27; it is divisible by 35, and $n^2 \equiv 1 \pmod{27}$, so $n \equiv \pm 1 \pmod{27}$. If $n = 35k$, this means $k \equiv \pm 17 \pmod{27}$. We have the obvious restrictions $35k < 730$ and $k \equiv 1 \pmod 2$, so $k = 17$ is the only option. Thus, $n = 35 \cdot 17 = \boxed{595}$.

7. Nine fair coins are flipped, and the coins are randomly divided into three piles of three coins. What is the expected number of heads in the pile with the largest number of heads?

   Proposed by Holden Mui

   Solution: First note that the final division is a smokescreen; we instead divide up the coins, *then* flip them. This is much more tractable.

   Let p_n denote the probability that the maximum number of coins a pile has is n. Then

   - p_0 is evidently $\left(\frac{1}{8}\right)^3 = \frac{1}{512}$,
   - p_1 is the probability that all piles have at most one head, minus p_0, id est $\left(\frac{4}{8}\right)^3 - \left(\frac{1}{8}\right)^3 = \frac{63}{512}$.
   - p_2 is the probability that that all piles have at most two heads, minus the probability that all piles have at most one head, which is $\left(\frac{7}{8}\right)^3 - \left(\frac{4}{8}\right)^3 = \frac{279}{512}$.
   - p_3 is equal to $1 - (p_0 + p_1 + p_2)$, or $\frac{169}{512}$.

   The expected largest number of heads is thus $p_1 + 2p_2 + 3p_3 = \boxed{\frac{141}{64}}$.

8. What is the sixth smallest positive real x such that $x \cdot \lfloor x \rfloor \cdot \{x\} = 6$? (Here, $\lfloor x \rfloor$ and $\{x\}$ denote the integer and fractional parts of x, respectively.)

   Proposed by Sean Li

   Solution: Denote $\lfloor x \rfloor = a$ and $\{x\} = b$ for sake of brevity. Then we want to solve $ab(a + b) = 6$. Firstly, note that $6 = ab(a + b) < a(1)(a + 1)$, so $a > 2$. We rewrite the equation as a quadratic in b and solve, yielding

   $$ab^2 + a^2b - 6 = 0 \implies b = \frac{-a^2 + \sqrt{a^4 + 24a}}{2a},$$

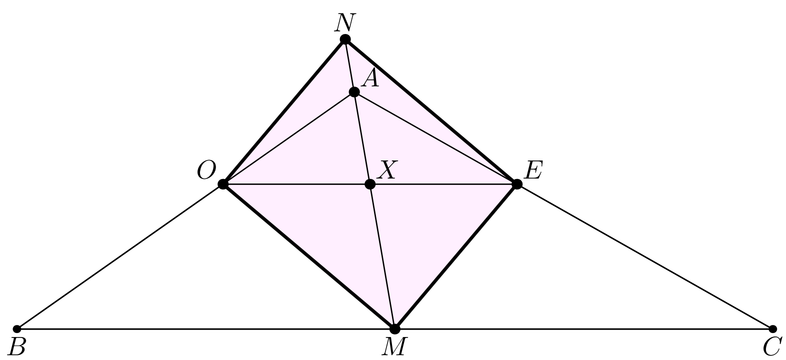
   id est $x = a + b = \frac{a^2+\sqrt{a^4+24a}}{2a}$. This gives us valid solutions for x for all integers $a > 2$, so the sixth smallest positive x occurs when $a = 8$, yielding $x = \frac{8^2+\sqrt{8^4+24 \cdot 8}}{2 \cdot 8} = \boxed{\frac{8+\sqrt{67}}{2}}$.

9. Triangle ABC has $AB = 6$, $BC = 11$, and $CA = 7$. Let M be the midpoint of $\overline{BC}$. Points E and O are on $\overline{AC}$ and $\overline{AB}$, respectively, and point N lies on line AM. Given that quadrilateral $NEMO$ is a rectangle, find its area.

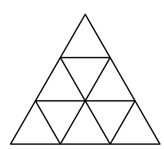
   Proposed by Sean Li

   Solution: First calculate $AM = \sqrt{\frac{2AB^2+2AC^2-BC^2}{4}} = \frac{7}{2}$. Let X be the center of $NEMO$. Then X is the midpoint of $\overline{EO}$ and lies on the median $\overline{AM}$, so $\overline{EO} \parallel \overline{BC}$. Thus, $\triangle AOE \sim \triangle ABC$, and $AX/AM = OE/BC$. But $OE = 2 \cdot OX = 2 \cdot MX = 2 \cdot (AM - AX)$, so $\frac{AX}{7/2} = \frac{7-2AX}{11}$, or $AX = 49/36$ and $MN = OE = 2 \cdot (7/2 - 49/36) = 77/18$.

   Finally, let $\theta = \angle AMC = \angle(\overline{MN}, \overline{OE})$. The height from A to $\overline{BC}$ is $\frac{12\sqrt{10}}{11}$, so $\sin\theta = \frac{24\sqrt{10}}{77}$ and the area of $NEMO$ is $\frac{1}{2}(MN)(EO)\sin\theta = \boxed{\frac{77\sqrt{10}}{27}}$.



10. Daniel paints each of the nine triangles in the diagram either crimson, scarlet, or maroon. Given that any pair of triangles sharing a side are painted different colors, how many ways can Daniel paint the diagram? Two diagrams that differ by a rotation or reflection are considered distinct.



   Proposed by Sean Li

   Solution: This problem can be solved with casework, but we provide a recursive solution (which can be generalized to more than three colors).

   We first color the six central triangles. The triangles form a cycle of length six, and we want to color the triangles such that no two adjacent triangles in the cycle are the same color.

   Let $f(n)$ denote the number of ways to color an n-cycle in this fashion. Then consider the following process to color the n-cycle: we first color a single vertex one of three colors, then move clockwise and color each vertex with a different color than the previous vertex. Then the last vertex is either the same color as the first vertex, which is equivalent to coloring an $(n-1)$-cycle and duplicating the last vertex, or a different color from the first vertex, which forms a valid n-cycle coloring. Thus

$$3 \cdot 2^{n-1} = f(n-1) + f(n).$$

   We have an initial value of $f(2) = 6$, so we compute $f(3) = 6, f(4) = 18, f(5) = 30, f(6) = 66$.

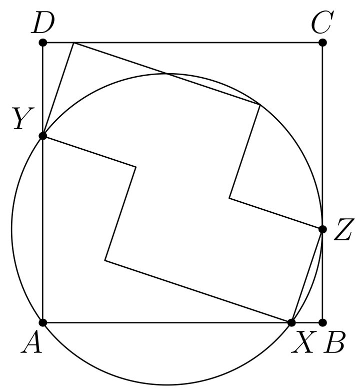
   Finally, we color the three corner triangles, each of which can be one of two colors. Our final answer is $66 \cdot 2^3 = \boxed{528}$.

   Remark. In this fashion, we get that the answer is $(n-1)^9 + (n-1)^3$ for n colors.

11. An S-tetromino is inscribed in square $ABCD$ such that their perimeters share exactly four points, one of which is X. Given that A, B, and X are collinear, maximize $\frac{AX}{BX}$. (An *S-tetromino* is a geometric figure comprised of four unit squares, joined edge to edge in the pictured fashion.)

Proposed by Holden Mui

Solution: Label points Y and Z as shown in the diagram.



Since $\angle XAY = \angle XZY = 90^\circ$, $XAYZ$ is cyclic; therefore $\angle XAZ = \angle XYZ$, implying $\triangle ABZ \sim \triangle YZX$, and as a result, $AB = 3 \cdot BZ$.

Hence, $AZ = YZ$ so the circle is tangent to $\overline{BC}$, which means $\angle BZX = \angle ZYX$, from which it follows that $\triangle BZX \sim \triangle ZYX$, so $BZ = 3 \cdot AB$.

Therefore, $AB = 3 \cdot BZ = 9 \cdot BX$, so $\frac{AX}{BX} = \boxed{8}$.

12. Let n be a positive integer. Points A, B, C are selected from the unit n-dimensional hypercube

$$\mathcal{H} = \{(x_1, x_2, \ldots, x_n) \mid 0 \le x_i \le 1 \text{ for } i = 1, \ldots, n\}.$$

Given that the maximum possible area of $\triangle ABC$ is a positive integer, minimize n.

Proposed by Holden Mui

Solution: Set the displacement vectors $\overrightarrow{AB} = (b_1, b_2, \ldots, b_n)$ and $\overrightarrow{AC} = (c_1, c_2, \ldots, c_n)$. Let $\angle BAC = \theta$; recall that

$$\cos\theta = \frac{b_1c_1 + \cdots + b_nc_n}{\sqrt{b_1^2 + \cdots + b_n^2} \cdot \sqrt{c_1^2 + \cdots + c_n^2}}$$

and $\sin\theta = \sqrt{1 - \cos^2\theta}$. The area of $\triangle ABC$ is thus $\frac{1}{2}(AB)(AC)\sin\theta$, or

$$\mathcal{A}_n = \frac{1}{2}\sqrt{(b_1^2 + \cdots + b_n^2)(c_1^2 + \cdots + c_n^2) - (b_1c_1 + \cdots + b_nc_n)^2}.$$

It is now clear by smoothing that $b_i, c_i \in \{0, 1\}$. Let $b_1 + \cdots + b_n = r$ and $c_1 + \cdots + c_n = s$; then

$$\max \mathcal{A}_n = \begin{cases} \frac{1}{2}\sqrt{rs - (r+s-n)^2} & \text{if } r + s > n, \\ \frac{1}{2}\sqrt{rs} & \text{otherwise} \end{cases}.$$

Let M_n denote the maximum possible value of the quantity within the square root; we wish to find the least n for which M_n is a (positive) even perfect square. It is a finite case check to show

$$M_2 = 1,\ M_3 = 3,\ M_4 = 5,\ M_5 = 8,\ M_6 = 12,$$

and $M_7 = 16$, so the answer is $n = \boxed{7}$.

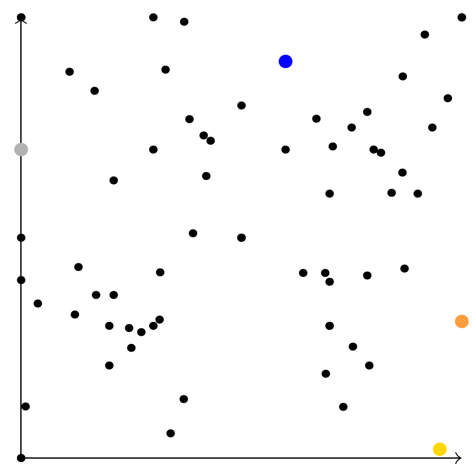
Remark. Actually, one can prove that $M_n = \lfloor n^2/3 \rfloor$, so $\max \mathcal{A}_n = \frac{1}{2}\sqrt{\lfloor n^2/3 \rfloor}$. The next two integer values are $\max \mathcal{A}_{97} = 28$ and $\max \mathcal{A}_{1351} = 390$; this sequence is `A011944`.

13. Pick a point $P = (x, y)$ such that $0 \leq x, y \leq 1$. Let S be the set of points submitted by all teams, $f(A)$ be the minimum distance from A to any other point in S, and let D be the maximum value of $f(A)$ across all points A in S. Your multiplier will be $0.9 + 0.01 \times \left\lfloor \frac{20f(P)}{D} \right\rfloor$. If you submit nothing or pick an invalid point, your multiplier will default to 0.9. In particular, you would like your point to be as far away from the closest other submitted point as possible.

Proposed by Benson Lin Zhan Li

Solution: Congratulations to the following teams for achieving a multiplier ≥ 1:

- Team **YEA Hong** from Youth EUCLID, who achieved $f((0.6, 0.9)) = 0.1414$ and a multiplier of 1.01,
- Team **Crystal Math** from Stuyvesant HS, who achieved $f((1, 0.31)) = 0.1769$ and a multiplier of 1.04,
- Team **Unlucky** from Stuyvesant HS, who achieved $f((0, 0.7)) = 0.2$ and a multiplier of 1.06, and
- Team **aaaaarikbqkazmxpdjdnw** from Stuyvesant HS, who achieved $f((0.95, 0.02)) = 0.2392$ and a multiplier of 1.1.



Here are some fun statistics:

- Twenty-six of the 100 teams submitted an invalid answer. Another 15 picked a point that another team chose (thus earning a multiplier of 0.9).
- The most commonly chosen point was $(0.5, 0.5)$, picked five times.

*NEMO 2019 was directed and run by Serena An, Eric Gan, Sean Li, Holden Mui, Ishika Shah, and Brandon Wang. We would like to thank Adithya Balachandran, Ankit Bisain, Benson Lin Zhan Li, Daniel Hong, Eric Shen, Luke Robitaille, Rishabh Das, and Sam Mitchell for submitting problem proposals. We would also like to thank Eric Shen, Daniel Hong, Luke Robitaille, and Rishabh Das for helping with problem selection and testsolving. Lastly, we would like to congratulate the **376** participants and **100** teams from **52** different schools for participating in this edition of NEMO.*