Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 20 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 20 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. Find the sum of the fourth smallest prime number and the fourth smallest composite number.

2. A daycare has babies that are one year, two years, and four years old. If the two-year-old babies are excluded, the average age of the remaining babies is 2.0; if the four-year-old babies are excluded, the average age of the remaining babies is 1.9; and if the one-year-old babies are excluded, the average age of the remaining babies is A. Compute A.

3. Andrew encounters a square $ABCD$, with side length one meter, on a sidewalk. He marks a point E such that B is on $\overline{DE}$ and $EB = \frac{1}{2}BD$. He then draws segments DE, EA, and AC with chalk, thus forming a giant four. Compute the total length, in meters, that Andrew has drawn.

4. Real numbers x, y, z satisfy the inequalities

$$-8 < x < 5, \quad -2 < y < 3, \quad -5 < z < 6.$$

There exist real numbers m and n such that $m < x \cdot y \cdot z < n$ for all choices of x, y, z. Find the minimum possible value of $n - m$.

5. Find the fifth smallest positive integer N such that the sum of the digits of N^2 is equal to the sum of the digits of N.

6. If $\dfrac{x}{y} = \dfrac{3}{x} = \dfrac{y}{4}$, compute the value of x^3.

7. Carol has a fair die with faces labeled 1 to 6. She rolls the die once, and if she rolls a 1, 2, or 3, she rolls the die a second time. What is the expected value of her last roll?

8. A *polyomino* is a geometric figure formed by joining one or more unit squares edge to edge. A *frame* is a polyomino with one hole such that the both its outside boundary and inside boundary are rectangles. Find the largest integer n for which there does not exist a frame with area n.

9. Let $ABCD$ be a square with center O, and let P be a randomly chosen point in the square's interior. What is the probability that triangles AOP, BOP, COP, and DOP are all obtuse?

10. Team MOP and Team MOSP are the last teams standing in a shouting tournament. The championship match consists of at most five rounds and will end when a team wins three rounds. Given that both teams have an equal chance of winning each round and that Team MOP won the fourth round, what is the probability that Team MOP won the tournament?

11. If n is a positive integer with d digits, let $p(n)$ denote the number of distinct d-digit numbers whose digits are a permutation of the digits of n. Find the smallest value of N such that $p(N) \geq 2019$.

12. Vincent chooses two (not necessarily distinct) positive divisors of 360 independently and at random. What is the probability that they are relatively prime?

13. If a and b are real numbers such that

$$\frac{a}{b} + \frac{b}{a} = 20 \quad \text{and} \quad \frac{a^2}{b} + \frac{b^2}{a} = 19,$$

then compute $|a - b|$.

14. Find all primes $p \geq 5$ such that p divides $(p-3)^{p-3} - (p-4)^{p-4}$.

15. The county of NEMO-landia has five towns, with no roads built between any two of them. How many ways can the NEMO-landian board build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

16. Externally tangent circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have radii 6 and 8, respectively. A line ℓ intersects $\mathcal{C}_1$ at A and B, and $\mathcal{C}_2$ at C and D. Given that $AB = BC = CD$, find AD.

17. Suppose x and y are positive reals satisfying
$$\sqrt{xy} = x - y = \frac{1}{x+y} = k.$$
Determine k.

18. Paper triangle ABC, with $\angle A > \angle B > \angle C$, has area 90. If side AB is folded in half, the area of the resulting figure is 54. If side BC is folded in half, the area of the resulting figure is 60. What is the area of the resulting figure if side CA is folded in half?

19. Find all reals $x > 1$ such that $\log_2\left(\log_2 x^8\right) \cdot \log_x\left(x^4 \log_x 2\right) = 12$.

20. Let ABC be an equilateral triangle and P a point in its interior obeying $AP = 5$, $BP = 7$, and $CP = 8$. Line CP intersects $\overline{AB}$ at Q. Compute AQ.