Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 20 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 20 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. If 5 bananas are worth the same amount as 6 apples, 4 apples are worth the same amount as 5 pears, and a pear costs \$2, what is the largest number of pieces of fruit Ryan can buy for \$12.96?

2. In regular hexagon $ABCDEF$, let M and N be the midpoints of sides AB and BC. Let P be the second intersection point of the circles centered at M and N that pass through B. What is the measure (in degrees) of $\angle MPN$?

3. Consider all 4! permutations of the letters in "GRUB", listed alphabetically. What is the sixth one? Express your answer as a four letter string.

4. Bob tells Alice that his favorite prime number leaves a remainder of 2 when divided by 5 and a remainder of 5 when divided by 11. He also tells Alice that it is less than 150. What is Bob's favorite prime number?

5. Moppers must mop at matching momentums (rates). My magical mats match my messy mezzanines in measure (size). $5M$ moppers mop my messy mezzanine in 9 minutes. Meanwhile, 18 moppers mop my mat in $90/M$ minutes. Manufacture the value of M.

6. Jeff is doing his homework. He may get distracted at 4pm, 5pm, and 6pm. At any time that he may get distracted, if he has already been distracted n times, the probability that he will get distracted is $\frac{1}{2^{n+1}}$. What is the probability that he will get distracted at 6pm?

7. Let $ABCD$ be a unit square, and let M and N be the midpoints of sides BC and CD, respectively. Let AM and AN meet BD at P and Q, respectively. Compute the area of quadrilateral $PQNM$.

8. Let $s(n)$ be the sum of the digits of n. Let $g(n) = (s(n))^2$. Find
$$g(g(\cdots(g(2018))\cdots)),$$
where g is applied 2018 times.

9. Anna, Bobby, Carol, and David are siblings. Their ages are all different positive integers and are greater than 5, while the sum of their ages is 55. Anna is the youngest, Bobby is the second youngest, Carol is the second oldest, and David is the oldest. What is the sum of all of Bobby's possible ages?

10. Let a *gang* be a nonempty set G of positive integers with the following properties:

    i) There exists an integer c, called the *pump* of G, such that for any a in G, there exists b in G such that $a + b = c$.

    ii) For every a, b in G, $\gcd(a, b) = 1$.

    A gang G is called *gucci* if there does not exist another gang H with the same constant c, such that G is a proper subset of H (that is, G is a subset of H but $G \neq H$). Let x be the number of distinct gucci gangs with pump 19, and let y be the number of distinct gucci gangs with pump 21.

    Find $10y + x$.

11. In a village of 2018 people (one of whom is Nemo), each day, 2 people are chosen at random. One is given an apple (and is not sacrificed), and one is sacrificed. What is the probability that Nemo never receives an apple? (The process ends when there is exactly 1 person left.)

12. Let $A_1A_2\ldots A_n$ be a regular n-gon with $n \geq 4$. Let ω_1 and ω_2 be circles with diameters A_1A_2 and A_3A_4, respectively, and let ω be the circle inside the n-gon tangent to ω_1, ω_2, and A_2A_3. If the centers of ω, ω_1, and ω_2 are collinear, what is n?

13. Let n be the largest positive integer that satisfies the following conditions:

    i) n has exactly 3 prime factors, all of which are single-digit,

    ii) n has no positive integer divisors of the form a^3b^2 for positive integers $a, b > 1$, and

    iii) n has no positive integer divisors of the form a^5b for positive integers $a, b > 1$.

    How many positive divisors does n have?

14. A teacher writes the alphabet (in order) on a board, and erases letters in a random order. What is the probability that at some point, `i` and `u` will be next to each other?

15. At a dinner, 50 mathematicians each order 1 of 5 possible entrees, and there is at least one order of each entree. The cost of entree n is equal to the number of people who order entree $n+1$, where entree 6 is entree 1. What is the minimum total cost of the dinner?

16. In triangle ABC with orthocenter H, let H' be the reflection of H across the perpendicular bisector of BC. If H' lies on line $\overline{AC}$, determine the largest possible value of the degree measure of $\angle C$. (The *orthocenter* of a triangle is defined to be the intersection of the 3 altitudes of the triangle.)

17. Let r be a real number such that $|r| < 1$ and $2 + 2r + 2r^2 + 2r^3 + \cdots = 2 + \sqrt{2}$. It is given that there is exactly one ordered pair (a, b) of positive integers such that $2r^b + 2r^{a+b} + 2r^{2a+b} + 2r^{3a+b} + \cdots = 1$. Compute $1000a + b$.

18. For each $1 \leq n \leq 10$, let a_n denote the number of ways to write 10 as the sum of n positive integers, where order matters (for example, $3 + 7$ and $7 + 3$ are considered different). Compute $\sum_{n=1}^{10} na_n$.

19. Let ABC be a triangle such that $\angle ABC = 82^\circ$ and $\angle ACB = 53^\circ$, and P be a point inside the triangle such that BCP is an isosceles right triangle with right angle at P. Compute $\angle APB$, in degrees.

20. Compute the smallest positive integer n such that there exists a five-digit positive integer S, whose decimal expansion is $a_1a_2a_3a_4a_5$, such that the decimal expansion of nS is $a_1000a_2a_3a_4a_5$.

Team Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 12 problems that are to be completed in 30 minutes, along with a small minigame that will generate a multiplier for a team's team round score.
- To ensure that your answers to problems 1 to 12 are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- The minigame, or problem 13, will ask your team to fill in an 8×8 grid satisfying certain conditions, and will generate a multiplier for your team round score. Be sure to try out the minigame, as failure to meet the conditions (even if nothing is submitted) may decrease your team's team round score.
- Each of the 12 team round problems have a predetermined point value; your team's team round score will be the sum of the point values assigned to each question that is correctly answered, multiplied by the multiplier generated by the minigame. Excluding the multiplier, a perfect score (achieved through answering all 12 problems correctly) is **400 points**.
- Your team score will be a combination of your score on the team round and the scores of each individual member.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your team name, team members' names, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. **[22]** Determine the smallest positive integer n such that $(n+1)^{n+2} - n^{n+1}$ is composite.

2. **[23]** Compute
$$\left\lfloor \frac{\binom{2018}{20}}{\binom{2018}{18}} \right\rfloor.$$

3. **[25]** For the mock PUMaC competition, Ben wants to assign two subjects out of algebra, combinatorics, geometry, and number theory to each of four team members such that each subject is assigned to exactly two members. How many ways are there to do this?

4. **[28]** In rectangle $ABCD$ with $AB = 8$ and $BC = 10$, cut four 1×1 squares off the corners, thus forming a 12-sided polygon with eight 90° angles and four 270° angles. To this polygon, add 4 quarter circles with radii 1, centers at the 270° angles, and oriented so that they don't overlap with the polygon. Compute the radius of the smallest circle that can circumscribe this new convex figure.

5. **[30]** Five real numbers are selected independently and at random from the interval $(0, 1)$. Let m be the minimum of the five selected numbers. What is the probability that $\frac{1}{4} < m < \frac{1}{2}$?

6. **[32]** In right triangle $\triangle ABC$ with $\angle BAC = 90^\circ$, the incircle ω touches BC, CA, and AB at D, E, and F, respectively. Let $G \neq E$ be the second intersection of BE with ω. If DG and AC are parallel and the radius of ω is 1, compute the area of $\triangle ABC$.

7. **[35]** Determine the minimum possible value of $(x+y)^2 - 2(x+2)(y+2)$ for real numbers x and y.

8. **[37]** How many ordered triples (x, y, z) of real numbers are there such that
$$xy + z = xz + y = yz + x = 0?$$

9. **[38]** For a positive integer n, let $d(n)$ be the number of divisors of n. Find the sum of all n such that
$$n = 7d(n).$$

10. **[42]** I am playing a 5 question trivia game in which I don't know any of the answers. However, the game gives me a list of the 5 correct answers, all distinct, to the questions in a random order. Every time I guess an answer to a question, I will find out whether or not I got it right before moving on to the next question. I guess the first answer on the list until I get a question correct, then I guess the second answer until I get another question correct, and so on, until I run out of questions. What is the expected value of the number of questions I will answer correctly?

11. **[43]** In square $ABCD$ with side length 6, let M and N denote the midpoints of BC and AD. A circle ω with radius 1 has center O contained on line MN. A tangent from A to ω and a tangent from B to ω intersect at a point P on line CD, so that ω lies inside ABP. What is $AP + BP$?

12. **[45]** Compute the sum of all fractions $0 < \frac{a}{b} < 1$ with a decimal representation that is periodic immediately after the decimal point with smallest period 6.

13. Let the number of people taking NEMO this year be an integer X. Estimate X. The multiplier to your team score will be $0.9 + 0.01 \left\lfloor 20 \min\left(\frac{X}{Y}, \frac{Y}{X}\right) \right\rfloor$ where Y is the answer you submit. (If your team submits nothing or a non-positive answer, your multiplier will default to 0.9.)

Answers

Individual Round

1. 6
2. 120
3. BURG
4. 137
5. 6
6. $\frac{19}{64}$
7. $\frac{5}{24}$
8. 169
9. 99
10. 69
11. $\frac{1}{2}$
12. 8
13. 20
14. $\frac{1}{78}$
15. 95
16. 135
17. 2001
18. 2816
19. 106
20. 501

Team Round

1. 3
2. 10521
3. 90
4. 6
5. $\frac{211}{1024}$
6. 6
7. -16
8. 5
9. 140
10. $\frac{103}{60}$
11. 18
12. 499455

Solutions

Individual Round

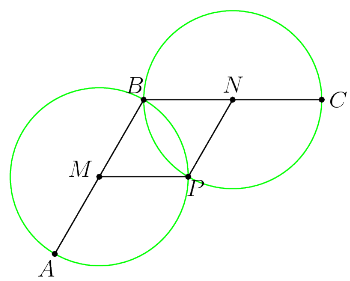
1. If 5 bananas are worth the same amount as 6 apples, 4 apples are worth the same amount as 5 pears, and a pear costs \$2, what is the largest number of pieces of fruit Ryan can buy for \$12.96?

   Proposed by Carl Schildkraut

   Solution: We can calculate that each apple costs \$2.50 and each banana costs \$3.00. So, to maximize the number of pieces of fruit Ryan can buy, he should only buy pears, of which he can afford $\boxed{6}$.

2. In regular hexagon $ABCDEF$, let M and N be the midpoints of sides AB and BC. Let P be the second intersection point of the circles centered at M and N that pass through B. What is the measure (in degrees) of $\angle MPN$?

   Proposed by Ishika Shah



   Solution: From the circles, $MP = MB$ and $NP = NB$, and $MB = NB$ since $AB = CB$. So $MBNP$ is a rhombus and $\angle MPN = \angle MBN = \boxed{120^\circ}$.

3. Consider all 4! permutations of the letters in “GRUB”, listed alphabetically. What is the sixth one? Express your answer as a four letter string.

   Proposed by Brandon Wang

   Solution: Note that since there are $3! = 6$ permutations that begin with the letter ”B”, the 6th one is the last one starting with ”B”, or $\boxed{\text{BURG}}$.

4. Bob tells Alice that his favorite prime number leaves a remainder of 2 when divided by 5 and a remainder of 5 when divided by 11. He also tells Alice that it is less than 150. What is Bob’s favorite prime number?

   Proposed by Carl Schildkraut

   Solution: Because it leaves a remainder of 2 (mod 5) and 5 (mod 11), it must be equivalent to 27 (mod 55) (this can be easily seen by noting that both 2 (mod 5) and 5 (mod 11) are -2^{-1}). The first two such integers, 27 and 82, are not prime, while the third such integer, $\boxed{137}$, is, giving us our answer.

5. Moppers must mop at matching momentums (rates). My magical mats match my messy mezzanines in measure (size). $5M$ moppers mop my messy mezzanine in 9 minutes. Meanwhile, 18 moppers mop my mat in $90/M$ minutes. Manufacture the value of M.

   Proposed by Ankit Bisain

Solution: Since it takes $5M$ people 9 minutes to complete a task, and 18 people and $\frac{90}{M}$ minutes to complete the same task, we get $5M(9) = 18\left(\frac{90}{M}\right) \implies M^2 = 36$ which means, since $M > 0$, we must have $M = \boxed{6}$.

6. Jeff is doing his homework. He may get distracted at 4pm, 5pm, and 6pm. At any time that he may get distracted, if he has already been distracted n times, the probability that he will get distracted is $\frac{1}{2^{n+1}}$. What is the probability that he will get distracted at 6pm?

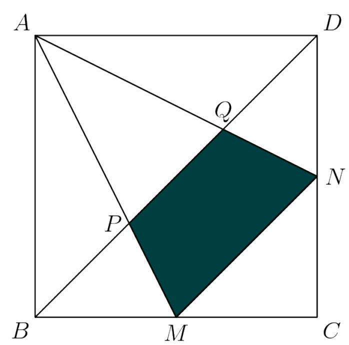
Proposed by Emily Wen

Solution: The probabilities that Jeff gets distracted only at 6pm, gets distracted at 4pm and 6pm, gets distracted at 5pm and 6pm, and gets distracted at all 3 times are $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$, $\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{32}$, $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{16}$, and $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} = \frac{1}{64}$, respectively. Therefore, the probability that he will get distracted at 6pm is

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} + \frac{1}{64} = \boxed{\frac{19}{64}}.$$

7. Let $ABCD$ be a unit square, and let M and N be the midpoints of sides BC and CD, respectively. Let AM and AN meet BD at P and Q, respectively. Compute the area of quadrilateral $PQNM$.

Proposed by Brandon Wang



Solution: Through angle chasing we find that $\triangle APD \sim \triangle MPB$ and $\triangle AQB \sim \triangle NQD$ with ratio $2:1$; therefore $\frac{AP}{AM} = \frac{AQ}{AN} = \frac{2}{2+1} = \frac{2}{3}$ so $[APQ] = \left(\frac{2}{3}\right)^2 [AMN] = \frac{4}{9}[AMN]$, or $[PQNM] = \frac{5}{9}[AMN]$. Therefore, since the area of $[AMN]$ is $1 - [ABM] - [ADN] - [MCN] = 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{8} = \frac{3}{8}$, the area of $PQNM$ is $\frac{5}{9} \times \frac{3}{8} = \boxed{\frac{5}{24}}$.

8. Let $s(n)$ be the sum of the digits of n. Let $g(n) = (s(n))^2$. Find

$$g(g(\cdots(g(2018))\cdots)),$$

where g is applied 2018 times.

Proposed by Brandon Wang and Eric Gan

Solution: Note that $g(g(g(g(2018)))) = g(g(g(121))) = g(g(16)) = g(49) = g(169)$. Now, since $g(g(169)) = g(256) = 169$, we therefore have $g^{2018}(2018) = g^{2014}(169) = g^{2\times 1007}(169) = \boxed{169}$, where $g^n(x)$ is g applied n times.

9. Anna, Bobby, Carol, and David are siblings. Their ages are all different positive integers and are greater than 5, while the sum of their ages is 55. Anna is the youngest, Bobby is the second youngest, Carol is the second oldest, and David is the oldest. What is the sum of all of Bobby's possible ages?

   Proposed by Sean Li

   Solution: Let the siblings' ages be $5 + a, 5 + b, 5 + c, 5 + d$, respectively, for some positive integers a, b, c, d. We are given that $a + b + c + d = 35$ and that they are all pairwise distinct.

   We compute the minimum and maximum ages of Bobby, so we find the extrema for b. Notice that $b \geq 2$, with equality when $c+d = 32$ (i.e. at $(1, 2, 15, 17)$). Meanwhile, $35 = a+b+c+d \geq 1+b+(b+1)+(b+2)$, implying that $b \leq 10$, with equality at $(1, 10, 11, 13)$.

   So Bobby's age is bounded between $2+5 = 7$ and $10+5 = 15$. It can be shown that all values between 7 and 15 are achievable, so our answer is $7 + 8 + \cdots + 15 = \boxed{99}$.

10. Let a *gang* be a nonempty set G of positive integers with the following properties:

    i) There exists an integer c, called the *pump* of G, such that for any a in G, there exists b in G such that $a + b = c$.

    ii) For every a, b in G, $\gcd(a, b) = 1$.

    A gang G is called *gucci* if there does not exist another gang H with the same constant c, such that G is a proper subset of H (that is, G is a subset of H but $G \neq H$). Let x be the number of distinct gucci gangs with pump 19, and let y be the number of distinct gucci gangs with pump 21.

    Find $10y + x$.

    Proposed by Yuru Niu

    Solution: There can be at most 1 pair of integers that add up to c in both of the gangs, because if there were more, both pairs of integers must have an even number and then not all integers will be relatively prime. Thus, each gang can be represented as $\{a, b\}$ with $a < b$.

    We first note that $x = 9$, because 19 is prime, so a and b have no other restrictions.

    Then, for y, the only restrictions on a and b are that they each must be relatively prime to 21. Therefore a can be 1, 2, 4, 5, 8, or 10, so $y = 6$.

    Therefore $10y + x = \boxed{69}$.

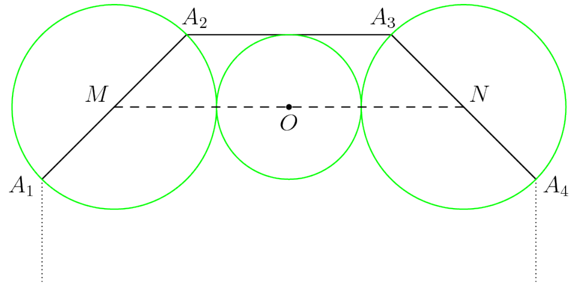
11. In a village of 2018 people (one of whom is Nemo), each day, 2 people are chosen at random. One is given an apple (and is not sacrificed), and one is sacrificed. What is the probability that Nemo never receives an apple? (The process ends when there is exactly 1 person left.)

    Proposed by Ankit Bisain

    Solution: Consider the first time Nemo is one of the 2 selected people. Nemo will either get sacrificed or get an apple, with probability $\frac{1}{2}$ each, so the answer is $\boxed{\dfrac{1}{2}}$.

12. Let $A_1A_2 \ldots A_n$ be a regular n-gon with $n \geq 4$. Let ω_1 and ω_2 be circles with diameters A_1A_2 and A_3A_4, respectively, and let ω be the circle inside the n-gon tangent to ω_1, ω_2, and A_2A_3. If the centers of ω, ω_1, and ω_2 are collinear, what is n?

    Proposed by Ishika Shah



Solution: Let M and N be the midpoints of A_1A_2 and A_3A_4, and O be the center of the third circle. WLOG let $A_1A_2 = 2$. From the collinear centers, $MN = 2 + 2r$, where r is the radius of the third circle. From isosceles trapezoid MA_2A_3N, $\angle A_2MN = \angle A_3NM = \frac{360}{n}$. So dropping perpendiculars from A_2 and A_3 to MN, we see $MN = 2\cos\frac{360}{n} + 2$. Then $r = \cos\frac{360}{n}$. But dropping a perpendicular from O to A_2A_3, $r = \sin\frac{360}{n}$, so $\sin\frac{360}{n} = \cos\frac{360}{n} \implies \frac{360}{n} = 45$ and so $n = \boxed{8}$.

13. Let n be the largest positive integer that satisfies the following conditions:

    i) n has exactly 3 prime factors, all of which are single-digit,
    ii) n has no positive integer divisors of the form a^3b^2 for positive integers $a, b > 1$, and
    iii) n has no positive integer divisors of the form a^5b for positive integers $a, b > 1$.

    How many positive divisors does n have?

    Proposed by Carl Schildkraut

    Solution: We have that $n = 2^a3^b5^c7^d$, where exactly one of $\{a, b, c, d\}$ is 0. We can without loss of generality state that $a \le b \le c \le d$, so $a = 0$ and $b, c, d > 0$.

    Case 1: $d \ge 3$. Here, by condition 2, we must have $b = c = 1$. However, condition 3 gives us that $d = 3$ or $d = 4$. So, the largest possible value of n in this case is $n = 3 \cdot 5 \cdot 7^4$.

    Case 2: $d \le 2$. Here, we cannot get anything contradicting conditions 2 or 3, so we might as well have $b = c = d = 2$, which gives $n = 3^2 \cdot 5^2 \cdot 7^2 = 105^2$. This is $\frac{15}{49}$ times the number we got in Case 1, so Case 1 gives the maximal n.

    For the answer extraction, we get $(1 + 1)(1 + 1)(4 + 1) = \boxed{20}$.

14. A teacher writes the alphabet (in order) on a board, and erases letters in a random order. What is the probability that at some point, `i` and `u` will be next to each other?

    Proposed by Ankit Bisain

    Solution: `i` and `u` cannot be removed before `j,k,l,m,n,o,p,q,r,s,t`, so only considering the order in which these 13 letters are removed, the answer is $\dfrac{11!2!}{13!} = \boxed{\dfrac{1}{78}}$.

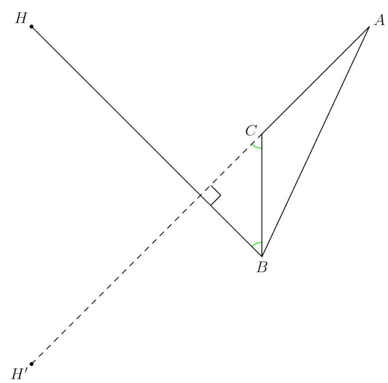
15. At a dinner, 50 mathematicians each order 1 of 5 possible entrees, and there is at least one order of each entree. The cost of entree n is equal to the number of people who order entree $n+1$, where entree 6 is entree 1. What is the minimum total cost of the dinner?

    Proposed by Ishika Shah

    Solution: The sum of the costs of the 5 entrees is \$50 since each person adds \$1 to the cost of exactly 1 of the entrees. Therefore, the total cost of the first order of each entree is \$50. Then each of the remaining 45 entrees must cost at least \$1 since at least one person ordered each entree. Therefore, the minimum total cost is $\boxed{\$95}$. One way this is achieved is when there is 1 order of each of the first 4 entrees and the remaining 46 people all order the last entree.

16. In triangle ABC with orthocenter H, let H' be the reflection of H across the perpendicular bisector of BC. If H' lies on line $\overline{AC}$, determine the largest possible value of the degree measure of $\angle C$. (The *orthocenter* of a triangle is defined to be the intersection of the 3 altitudes of the triangle.)

    Proposed by Jeffery Li

    Solution: Assume $\angle C$ is obtuse, since we are trying to find the largest possible degree measure. Then $\angle H'CB = \angle HBC = \angle BCA - 90^\circ$. However, since H' lies on AC, we have $\angle H'CB = 180^\circ - \angle BCA$. Equating the two and solving, we get $\angle BCA = \boxed{135^\circ}$. No higher degree measures work, so this is the largest.

17. Let r be a real number such that $|r| < 1$ and $2 + 2r + 2r^2 + 2r^3 + \cdots = 2 + \sqrt{2}$. It is given that there is exactly one ordered pair (a, b) of positive integers such that $2r^b + 2r^{a+b} + 2r^{2a+b} + 2r^{3a+b} + \cdots = 1$. Compute $1000a + b$.

    Proposed by Jeffery Li

    Solution: The given condition implies $\frac{2}{1-r} = 2 + \sqrt{2}$, or $r = 1 - \frac{2}{2+\sqrt{2}} = 1 - (2 - \sqrt{2}) = \sqrt{2} - 1$. Now, we want to find (a, b) such that $\frac{2r^b}{1-r^a} = 1$, or $2r^b + r^a = 1$. Since a and b are positive integers, r^a, $r^b \leq r = \sqrt{2} - 1 < 1.5 - 1 = 0.5$. Therefore, if $b \geq 2$, then $r^b \leq 3 - 2\sqrt{2} < 3 - 2.8 = 0.2$, meaning that $2r^b + r^a < 2(0.2) + 0.5 = 0.9 < 1$, contradiction. Thus, $b = 1$, so $r^a = 1 - 2r = 1 - 2(\sqrt{2} - 1) = 3 - 2\sqrt{2}$, or $a = 2$. Therefore, our ordered pair is $(2, 1)$, and the desired answer is $1000(2) + 1 = \boxed{2001}$.

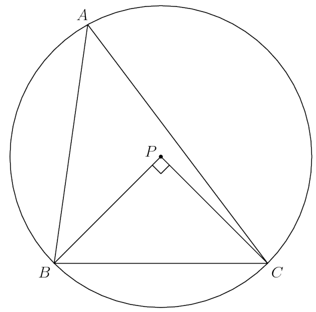
18. For each $1 \leq n \leq 10$, let a_n denote the number of ways to write 10 as the sum of n positive integers, where order matters (for example, $3 + 7$ and $7 + 3$ are considered different). Compute $\sum_{n=1}^{10} na_n$.

    Proposed by Ben Qi

    Solution: Consider a row of ten ones, with nine possible division points. On average, each partition contains $\frac{1+10}{2}$ elements and there are 2^9 in total, each corresponding to a way to write 10 as the sum of positive integers, so our answer is
    $$\frac{11}{2} \cdot 2^9 = \boxed{2816}.$$

19. Let ABC be a triangle such that $\angle ABC = 82^\circ$ and $\angle ACB = 53^\circ$, and P be a point inside the triangle such that BCP is an isosceles right triangle with right angle at P. Compute $\angle APB$, in degrees.

Proposed by Holden Mui



Solution: Note that P is the circumcenter of ABC because $\angle BPC = 2\angle BAC$ and $BP = CP$, and thus $\angle APB = 2\angle ACB = \boxed{106^\circ}$.

20. Compute the smallest positive integer n such that there exists a five-digit positive integer S, whose decimal expansion is $a_1a_2a_3a_4a_5$, such that the decimal expansion of nS is $a_1000a_2a_3a_4a_5$.

Proposed by Jeffery Li

Solution: Let $X = a_1 \times 10^4$ and $Y = S - X = a_2a_3a_4a_5$. Note that $X > Y$. Now, that we essentially want the smallest n such that it's possible to have $\frac{1000X+Y}{X+Y} = n$, or $(1000 - n)X = (n - 1)Y$. Since $X > Y$ we must have $n - 1 > 1000 - n$, or $n > 500.5$, or $n \geq 501$. Now, for $n = 501$, we note that $S = 19980$ works, as $nS = (501)(19980) = 10009980$. Therefore, the smallest such positive integer is $\boxed{501}$.

Team Round

1. Determine the smallest positive integer n such that $(n+1)^{n+2} - n^{n+1}$ is composite.

   Proposed by Jeffery Li

   Solution: Clearly $2^3 - 1^2 = 8 - 1 = 7$ and $3^4 - 2^3 = 81 - 8 = 73$ are prime, so $n \geq 3$. Now, note that $4^5 - 3^4 = 32^2 - 9^2 = 41 \times 23$ is composite, so $n = \boxed{3}$ works.

2. Compute
$$\left\lfloor \frac{\binom{2018}{20}}{\binom{2018}{18}} \right\rfloor.$$

   Proposed by Alex Xu

   Solution: Note that
$$\left\lfloor \frac{\binom{2018}{20}}{\binom{2018}{18}} \right\rfloor = \left\lfloor \frac{\frac{2018!}{20!1998!}}{\frac{2018!}{18!2000!}} \right\rfloor = \left\lfloor \frac{2000!18!}{1998!20!} \right\rfloor = \left\lfloor \frac{1999 \times 2000}{19 \times 20} \right\rfloor = \left\lfloor \frac{199900}{19} \right\rfloor = \boxed{10521}.$$

3. For the mock PUMaC competition, Ben wants to assign two subjects out of algebra, combinatorics, geometry, and number theory to each of four team members such that each subject is assigned to exactly two members. How many ways are there to do this?

   Proposed by Ben Qi

   Solution: We will use casework.

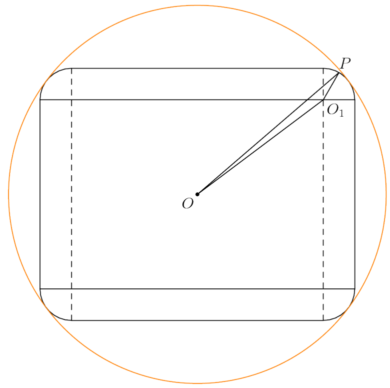
   Case 1: There exist two team members with the exact same subjects. Note that there are three ways to partition the four subjects into two groups of two, and then $\binom{4}{2}$ ways to assign the groups to members, so our count is
$$3 \cdot \binom{4}{2} = 18.$$

   Case 2: There do not exist two team members with the exact same subjects. First we assign two random subjects to the first person. Then there are exactly $2 \cdot 3!$ ways to assign subjects to the remaining members, so our count is
$$\binom{4}{2} \cdot 2 \cdot 3! = 72.$$

   Our final answer is $18 + 72 = \boxed{90}$.

4. In rectangle $ABCD$ with $AB = 8$ and $BC = 10$, cut four 1×1 squares off the corners, thus forming a 12-sided polygon with eight 90° angles and four 270° angles. To this polygon, add 4 quarter circles with radii 1, centers at the 270° angles, and oriented so that they don't overlap with the polygon. Compute the radius of the smallest circle that can circumscribe this new convex figure.

   Proposed by Jeffery Li



Solution: Clearly the smallest circle has a center O at the center of symmetry of the figure. Let P be a point in this new figure. We want to choose P such that OP is maximized. It's clear that P is on the boundary of this figure and on one of the four circular arcs (else we can move P along the boundary towards the arcs).

Let O_1 be the center of one of the circular arcs. Now, by Pythagorean Theorem, $OO_1 = \sqrt{3^2+4^2} = 5$, and by Triangle Inequality, $OP \le OO_1 + O_1P = 5+1 = 6$, where equality holds if O, O_1, P are collinear. Thus, the smallest circle that can circumscribe this figure is the maximum length of OP, which is $\boxed{6}$.

5. Five real numbers are selected independently and at random from the interval $(0,1)$. Let m be the minimum of the five selected numbers. What is the probability that $\frac{1}{4} < m < \frac{1}{2}$?

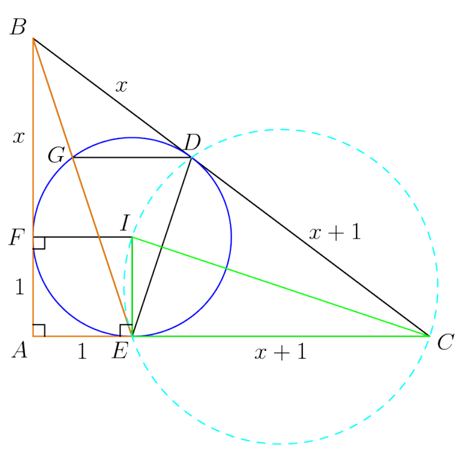
   Proposed by Sean Li

   Solution: The probability in question is equivalent to

$$P\left(\text{all numbers are} \ge \frac{1}{4}\right) - P\left(\text{all numbers are} \ge \frac{1}{2}\right) = \left(\frac{3}{4}\right)^5 - \left(\frac{1}{2}\right)^5 = \frac{243}{1024} - \frac{1}{32} = \boxed{\frac{211}{1024}}.$$

6. In right triangle $\triangle ABC$ with $\angle BAC = 90^\circ$, the incircle ω touches BC, CA, and AB at D, E, and F, respectively. Let $G \neq E$ be the second intersection of BE with ω. If DG and AC are parallel and the radius of ω is 1, compute the area of $\triangle ABC$.

   Proposed by Ankit Bisain



Solution: Note that $AEIF$ is a square. So, $EI \perp AC \parallel EG$. Since $ID = IG$, the line through I perpendicular to DG is the perpendicular bisector of DG; E lies on this line and so $ED = EG$. So, EI bisects $\angle DEG$. Since $IE \perp AC \perp AB$, we have that $IE \parallel AB$, and so since $IECD$ is cyclic (all four points lie on the circle with diameter IC),

$$\angle ABE = \angle GEI = \angle DEI = \angle DCI = \angle ECI,$$

where the last equality follows from the fact that CI bisects $\angle BCA$. Now, since $AE = IE = 1$, by SAA congruence $\triangle ABE \cong \triangle ECI$. So, if we let $BF = BD = x$, we have

$$CD = CE = AB = x + 1.$$

Now, by the Pythagorean Theorem, $(x+1)^2 + (x+2)^2 = (2x+1)^2$, or $x = 2$ ($x = -1$ is extraneous), so we have a $3-4-5$ triangle with area $\boxed{6}$.

7. Determine the minimum possible value of $(x+y)^2 - 2(x+2)(y+2)$ for real numbers x and y.

Proposed by Holden Mui

Solution: The expression is equivalent to $(x-2)^2 + (y-2)^2 - 16$, so the minimum is $\boxed{-16}$.

8. How many ordered triples (x, y, z) of real numbers are there such that

$$xy + z = xz + y = yz + x = 0?$$

Proposed by Carl Schildkraut

Solution: The first condition gives $z = -xy$. Under this substitution, the next two give

$$x(1-y^2) = y(1-x^2) = 0.$$

Case 1: $x = 0$. Here, the second condition gives $y = 0$, for 1 value of (x, y, z).

Case 2: $1 = y^2$. Here, as $y \neq 0$, we must have $1 = x^2$. As each of x and y can vary in $\{-1, 1\}$, there are 4 possibilities in this case.

Thus, the number of ordered triples is $1 + 4 = \boxed{5}$.

9. For a positive integer n, let $d(n)$ be the number of divisors of n. Find the sum of all n such that

$$n = 7d(n).$$

Proposed by Jeremy Zhou

Solution: Consider the function $f(n) = \frac{n}{d(n)}$. Since $f(n)$ is multiplicative for coprime prime powers (p^a, q^b), we can consider each prime factor of n separately.

We can thus make a nice table of $f(n)$:

a	$f(2^a)$	$f(3^a)$	$f(5^a)$	$f(7^a)$
1	1	3/2	5/2	7/2
2	4/3	3	25/3	49/3
3	2	27/4	125/4	343/4
4	16/5	81/5	125	2401/5

We can only get a factor of 7 in $f(n)$ using the 7 column, and since $\frac{49}{3} > 7$ we must have $\frac{7}{2}$.

We now need a factor of 2. We can use 2 or $\frac{4}{3} \cdot \frac{3}{2}$, but everything else has too many factors of 2 and is thus unusable.

In conclusion, we have $f(n) = \frac{7}{2} \cdot 2$ or $\frac{7}{2} \cdot \frac{4}{3} \cdot \frac{3}{2}$, which correspond to $n = 7 \cdot 2^3 = 56$ and $n = 7 \cdot 2^2 \cdot 3 = 84$. Thus, the sum of all such n is $56 + 84 = \boxed{140}$.

10. I am playing a 5 question trivia game in which I don't know any of the answers. However, the game gives me a list of the 5 correct answers, all distinct, to the questions in a random order. Every time I guess an answer to a question, I will find out whether or not I got it right before moving on to the next question. I guess the first answer on the list until I get a question correct, then I guess the second answer until I get another question correct, and so on, until I run out of questions. What is the expected value of the number of questions I will answer correctly?

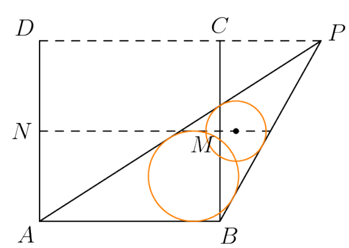
    Proposed by Ishika Shah

    Solution: Let the answer choices be `a, b, c, d` and `e` in that order. Let A be the question with answer `a`, B the question with answer `b`, and so on. I will always get question A correct. I will get a second question correct iff question A is asked before question B, which has probability $\frac{1}{2}$. I will get a third question correct iff questions A, B, and C are asked in that order, which has probability $\frac{1}{3!} = \frac{1}{6}$. Similarly, I will get 4 questions correct with probability $\frac{1}{24}$, and 5 questions correct with probability $\frac{1}{120}$. So the expected value of the number of correct answers is

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{206}{120} = \boxed{\frac{103}{60}}.$$

11. In square $ABCD$ with side length 6, let M and N denote the midpoints of BC and AD. A circle ω with radius 1 has center O contained on line MN. A tangent from A to ω and a tangent from B to ω intersect at a point P on line CD, so that ω lies inside ABP. What is $AP + BP$?

    Proposed by Ankit Bisain



Solution: Since the point on ω furthest away from CD is at a distance 4 away, using a dilation with center P and ratio $\frac{3}{2}$, we note that ABP has inradius $\frac{3}{2}$, so using $A = rs$, $\frac{3}{4}(6 + AP + BP) = \frac{6 \times 6}{2}$ which solves to $AP + BP = \boxed{18}$.

Remark: The original version of this problem (which was given out) had a typo with *segment CD* instead of *line CD*.

12. Compute the sum of all fractions $0 < \frac{a}{b} < 1$ with a decimal representation that is periodic immediately after the decimal point with smallest period 6.

    Proposed by Milan Haiman

Solution: Such fractions are of the form $\frac{a}{999999}$ but not of the form $\frac{a}{999}$ or $\frac{a}{99}$. Using inclusion-exclusion to count, we get $999999 - 999 - 99 + 9 = 998910$ such fractions. Grouping them into pairs with sum 1, we get that the sum is half the number of fractions, or $\frac{998910}{2} = \boxed{499455}$.

13. Let the number of people taking NEMO this year be an integer X. Estimate X. The multiplier to your team score will be $0.9 + 0.01 \left\lfloor 20 \min\left(\frac{X}{Y}, \frac{Y}{X}\right) \right\rfloor$ where Y is the answer you submit. (If your team submits nothing or a non-positive answer, your multiplier will default to 0.9.)

    Comments: For this year, X turned out to be $\boxed{230}$. 18 teams were at most a factor of 2 off and managed to get a multiplier of over 1. Congratulations to team "Came for Problems, Stayed for Chips" for achieving a multiplier of 1.09 with a guess of 227.

NEMO 2018 was directed and run by Brandon Wang, Jeffery Li, Ishika Shah, Sean Li, Eric Gan, and Jeremy Zhou. We would like to thank Alex Xu, Ankit Bisain, Ben Qi, Carl Schildkraut, Emily Wen, Holden Mui, Maxwell Jiang, Milan Haiman, and Yuru Niu for submitting problem proposals. We would also like to thank Ankit Bisain, Milan Haiman, and Tristan Shin for helping with problem selection and testsolving. Lastly, we would like to congratulate the 230 participants and 57 teams from 26 different schools for participating in this edition of NEMO.