Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 20 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 20 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. If 5 bananas are worth the same amount as 6 apples, 4 apples are worth the same amount as 5 pears, and a pear costs \$2, what is the largest number of pieces of fruit Ryan can buy for \$12.96?

2. In regular hexagon $ABCDEF$, let M and N be the midpoints of sides AB and BC. Let P be the second intersection point of the circles centered at M and N that pass through B. What is the measure (in degrees) of $\angle MPN$?

3. Consider all 4! permutations of the letters in "GRUB", listed alphabetically. What is the sixth one? Express your answer as a four letter string.

4. Bob tells Alice that his favorite prime number leaves a remainder of 2 when divided by 5 and a remainder of 5 when divided by 11. He also tells Alice that it is less than 150. What is Bob's favorite prime number?

5. Moppers must mop at matching momentums (rates). My magical mats match my messy mezzanines in measure (size). $5M$ moppers mop my messy mezzanine in 9 minutes. Meanwhile, 18 moppers mop my mat in $90/M$ minutes. Manufacture the value of M.

6. Jeff is doing his homework. He may get distracted at 4pm, 5pm, and 6pm. At any time that he may get distracted, if he has already been distracted n times, the probability that he will get distracted is $\frac{1}{2^{n+1}}$. What is the probability that he will get distracted at 6pm?

7. Let $ABCD$ be a unit square, and let M and N be the midpoints of sides BC and CD, respectively. Let AM and AN meet BD at P and Q, respectively. Compute the area of quadrilateral $PQNM$.

8. Let $s(n)$ be the sum of the digits of n. Let $g(n) = (s(n))^2$. Find
$$g(g(\cdots(g(2018))\cdots)),$$
where g is applied 2018 times.

9. Anna, Bobby, Carol, and David are siblings. Their ages are all different positive integers and are greater than 5, while the sum of their ages is 55. Anna is the youngest, Bobby is the second youngest, Carol is the second oldest, and David is the oldest. What is the sum of all of Bobby's possible ages?

10. Let a *gang* be a nonempty set G of positive integers with the following properties:
    i) There exists an integer c, called the *pump* of G, such that for any a in G, there exists b in G such that $a + b = c$.
    ii) For every a, b in G, $\gcd(a, b) = 1$.

    A gang G is called *gucci* if there does not exist another gang H with the same constant c, such that G is a proper subset of H (that is, G is a subset of H but $G \neq H$). Let x be the number of distinct gucci gangs with pump 19, and let y be the number of distinct gucci gangs with pump 21.

    Find $10y + x$.

11. In a village of 2018 people (one of whom is Nemo), each day, 2 people are chosen at random. One is given an apple (and is not sacrificed), and one is sacrificed. What is the probability that Nemo never receives an apple? (The process ends when there is exactly 1 person left.)

12. Let $A_1A_2\ldots A_n$ be a regular n-gon with $n \geq 4$. Let ω_1 and ω_2 be circles with diameters A_1A_2 and A_3A_4, respectively, and let ω be the circle inside the n-gon tangent to ω_1, ω_2, and A_2A_3. If the centers of ω, ω_1, and ω_2 are collinear, what is n?

13. Let n be the largest positive integer that satisfies the following conditions:

    i) n has exactly 3 prime factors, all of which are single-digit,

    ii) n has no positive integer divisors of the form a^3b^2 for positive integers $a, b > 1$, and

    iii) n has no positive integer divisors of the form a^5b for positive integers $a, b > 1$.

    How many positive divisors does n have?

14. A teacher writes the alphabet (in order) on a board, and erases letters in a random order. What is the probability that at some point, `i` and `u` will be next to each other?

15. At a dinner, 50 mathematicians each order 1 of 5 possible entrees, and there is at least one order of each entree. The cost of entree n is equal to the number of people who order entree $n+1$, where entree 6 is entree 1. What is the minimum total cost of the dinner?

16. In triangle ABC with orthocenter H, let H' be the reflection of H across the perpendicular bisector of BC. If H' lies on line $\overline{AC}$, determine the largest possible value of the degree measure of $\angle C$. (The *orthocenter* of a triangle is defined to be the intersection of the 3 altitudes of the triangle.)

17. Let r be a real number such that $|r| < 1$ and $2 + 2r + 2r^2 + 2r^3 + \cdots = 2 + \sqrt{2}$. It is given that there is exactly one ordered pair (a, b) of positive integers such that $2r^b + 2r^{a+b} + 2r^{2a+b} + 2r^{3a+b} + \cdots = 1$. Compute $1000a + b$.

18. For each $1 \le n \le 10$, let a_n denote the number of ways to write 10 as the sum of n positive integers, where order matters (for example, $3 + 7$ and $7 + 3$ are considered different). Compute $\sum_{n=1}^{10} na_n$.

19. Let ABC be a triangle such that $\angle ABC = 82^\circ$ and $\angle ACB = 53^\circ$, and P be a point inside the triangle such that BCP is an isosceles right triangle with right angle at P. Compute $\angle APB$, in degrees.

20. Compute the smallest positive integer n such that there exists a five-digit positive integer S, whose decimal expansion is $a_1a_2a_3a_4a_5$, such that the decimal expansion of nS is $a_1000a_2a_3a_4a_5$.