# Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 25 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than $10^{10}$). Write fractional answers in the form $\frac{a}{b}$ where $a, b$ are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 25 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. Square $ABCD$ is inscribed in circle $\omega_1$ of radius 4. Points $E$, $F$, $G$, and $H$ are the midpoints of sides $AB$, $BC$, $CD$, and $DA$, respectively, and circle $\omega_2$ is inscribed in square $EFGH$. The area inside $\omega_1$ but outside $ABCD$ is shaded, and the area inside $EFGH$ but outside $\omega_2$ is shaded. Compute the total area of the shaded regions.

2. In a very strange orchestra, the ratio of violins to violas to cellos to basses is $4\colon 5\colon 3\colon 2$. The volume of a single viola is twice that of a single bass, but $\frac{2}{3}$ that of a single violin. The combined volume of the cellos is equal to the combined volume of all the other instruments in the orchestra. How many basses are needed to match the volume of a single cello?

3. The 75 students at MOP are partitioned into 10 teams, all of distinct sizes. What is the greatest possible number of students in the fifth largest group? (A team may have only 1 person, but must have at least 1 person.)

4. Find all ordered 8-tuples of positive integers $(a, b, c, d, e, f, g, h)$ such that:
$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}+\frac{1}{e^3}+\frac{1}{f^3}+\frac{1}{g^3}+\frac{1}{h^3}=1.$$

5. Evaluate
$$\sum_{n=0}^{100}(-1)^n n^2.$$

6. Alan starts with the numbers 1, 2, . . ., 10 written on a blackboard. He chooses three of the numbers, and erases the second largest one of them. He repeats this process until there are only two numbers left on the board. What are all possible values for the sum of these two numbers?

7. The first three terms of an infinite geometric series are 1, $-x$, and $x^2$. The sum of the series is $x$. Compute $x$.

8. Regular hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in a circle with center $O$ and radius 2. 6 circles with diameters $OA_i$ for $1 \le i \le 6$ are drawn. Compute the area inside the circle with radius 2 but outside the 6 smaller circles.

9. Let $a$, $b$, $c$ be distinct integers from the set $\{3, 5, 7, 8\}$. What is the greatest possible value of the units digit of $a^{(b^c)}$?

10. I am playing a very curious computer game. Originally, the monitor displays the number 10, and I will win when the screen shows the number 0. Each time I press a button, the number onscreen will decrease by either 1 or 2 with equal probability, unless the number is 1, in which case it will always decrease to zero. What is the probability that it will take me exactly 8 button presses to win?

11. How many ways are there to rearrange the letters in "ishika shah" into two words so that the first word has six letters, the second word has four letters, and the first word has exactly five distinct letters? Note that letters can switch between words, and switching between two of the same letter does not produce a new arrangement.

12. Let
$$N = 2018^{\frac{2018}{1+\frac{1}{\log_2(1009)}}}.$$
$N$ can be written in the form $a^b$, where $a$ and $b$ are positive integers and $a$ is as small as possible. What is $a + b$?

13. Triangle $ABC$ has $AB = 17$, $BC = 25$, and $CA = 28$. Let $X$ be the point such that $AX \perp AC$ and $AC \parallel BX$. Let $Y$ be the point such that $BY \perp BC$ and $AY \parallel BC$. Find the area of $ABXY$.

14. Find the last two digits of $17^{(20^{17})}$ in base 9. (Here, the numbers given are in base 10.)

15. How many positive integers $n$ satisfy the following properties:

    (a) There exists an integer $a > 2017$ such that $\gcd(a, n) = 2017$.

    (b) There exists an integer $b < 1000$ such that $\operatorname{lcm}(b, n) = 2017000$.

16. Nathan is given the infinite series
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$
Starting from the leftmost term, Nathan will follow this procedure:

    (a) If removing the term does not cause the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will remove the term and move on to the next term.

    (b) If removing the term causes the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will not remove the term and move on to the next term.

    How many of the first 100 terms will he end up removing?

17. Find all positive integers $1 \le k \le 289$ such that $k^2 - 32$ is divisible by 289.

18. Daniel has a broken four-function calculator, on which only the buttons 2, 3, 5, 6, $+$, $-$, $\times$, and $\div$ work. Daniel plays the following game: First, he will type in one of the digits with equal probability, then one of the four operators with equal probability, then one of the digits with equal probability, and then he will let the calculator evaluate the expression. What is the expected value of the result he gets?

19. We call a pair of positive integers *quare* if their product is a perfect square. A subset $S$ of $\{1, 2, \ldots, 100\}$ is chosen, such that no two distinct elements of $S$ form a quare pair. Find the maximum possible numbers in such a set $S$.

20. Two circles of radii 20 and 18 are an excircle and incircle of a triangle $\mathcal{T}$. If the distance between their centers is 47, compute the area of $\mathcal{T}$.

21. Let $ABC$ be a triangle, $M$ be the midpoint of $BC$, and $D$ the foot of the altitude from $A$ to $BC$. Let $O$ be the circumcenter of $ABC$, and $H$ the orthocenter of $ABC$. If $OHDM$ is a square with side length 1, find $|AC - AB|$.

22. The value of the expression
$$\sqrt{1 + \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}} + \sqrt{1 - \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}}$$
can be written as $\sqrt[m]{n}$, where $m$ and $n$ are positive integers. Compute the smallest possible value of $m + n$.

23. Triangle $ABC$ has $\angle A = 73^\circ$, and $\angle C = 38^\circ$. Let $M$ be the midpoint of $AC$, and let $X$ be the reflection of $C$ over $BM$. If $E$ and $F$ are the feet of the altitudes from $A$ and $C$, respectively, then what is the measure, in degrees, of $\angle EXF$?

24. Compute the remainder when $\binom{3^4}{3^2}$ is divided by $3^8$.

25. Consider sets $S = \{s_1, s_2, \ldots, s_n\}$ of integers for which each of $\{1, 2, 4, 8, 16\}$ can be written as the sum of distinct elements of $S$. Find the minimum possible value of
$$500n + |s_1| + |s_2| + \cdots + |s_n|.$$

# Team Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 14 problems that are to be completed in 30 minutes, along with a small minigame that will generate a multiplier for a team's team round score.
- To ensure that your answers to problems 1 to 14 are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than $10^{10}$). Write fractional answers in the form $\frac{a}{b}$ where $a, b$ are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- The minigame, or problem 15, will ask your team to fill in an $8 \times 8$ grid satisfying certain conditions, and will generate a multiplier for your team round score. Be sure to try out the minigame, as failure to meet the conditions (even if nothing is submitted) may decrease your team's team round score.
- Each of the 14 team round problems have a predetermined point value; your team's team round score will be the sum of the point values assigned to each question that is correctly answered, multiplied by the multiplier generated by the minigame. Excluding the multiplier, a perfect score (achieved through answering all 14 problems correctly) is **400 points**.
- Your team score will be a combination of your score on the team round and the scores of each individual member.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your team name, team members' names, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. **[10]** A team of 6 distinguishable students competed at a math competition. They each scored an integer amount of points, and the sum of their scores was 170. Their highest score was a 29, and their lowest score was a 27. How many possible ordered 6-tuples of scores could they have scored?

2. **[10]** Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let $P$ be a point inside triangle $ABC$, and let ray $AP$ meet segment $BC$ at $Q$. Suppose the area of triangle $ABP$ is three times the area of triangle $CPQ$, and the area of triangle $ACP$ is three times the area of triangle $BPQ$. Compute the length of $BQ$.

3. **[10]** I am thinking of a geometric sequence with 9600 terms, $a_1$, $a_2$,. . ., $a_{9600}$. The sum of the terms with indices divisible by three (i.e. $a_3 + a_6 + \cdots + a_{9600}$) is $\frac{1}{56}$ times the sum of the other terms (i.e. $a_1 + a_2 + a_4 + a_5 + \cdots + a_{9598} + a_{9599}$). Given that the terms with even indices sum to 10, what is the smallest possible sum of the whole sequence?

4. **[15]** Let $ABCD$ be a regular tetrahedron with side length $6\sqrt{2}$. There is a sphere centered at each of the four vertices, with the radii of the four spheres forming a geometric series with common ratio 2 when arranged in increasing order. If the volume inside the tetrahedron but outside the second largest sphere is 71, what is the volume inside the tetrahedron but outside all four of the spheres?

5. **[20]** Find the largest number of consecutive positive integers, each of which has exactly 4 positive divisors.

6. **[25]** Let the (not necessarily distinct) roots of the equation $x^{12} - 3x^4 + 2 = 0$ be $a_1, a_2, \ldots, a_{12}$. Compute
$$\sum_{i=1}^{12} |\mathrm{Re}(a_i)|.$$

7. **[25]** Jeffrey is doing a three-step card trick with a row of seven cards labeled A through G. Before he starts his trick, he picks a random permutation of the cards. During each step of his trick, he rearranges the cards in the order of that permutation. For example, for the permutation $(1, 3, 5, 2, 4, 7, 6)$, the first card from the left remains in position, the second card is moved to the third position, the third card is moved to the fifth position, etc. After Jeffrey completes all three steps, what is the probability that the "A" card will be in the same position as where it started?

8. **[30]** In convex equilateral hexagon $ABCDEF$, $AC = 13$, $CE = 14$, and $EA = 15$. It is given that the area of $ABCDEF$ is twice the area of triangle $ACE$. Compute $AB$.

9. **[35]** Find all ordered pairs $(x, y)$ of numbers satisfying
$$(1 + x^2)(1 + y^2) = 170$$
$$(1 + x)(-1 + y) = 10.$$

10. **[40]** Call a positive integer "pretty good" if it is divisible by the product of its digits. Call a positive integer $n$ "clever" if $n$, $n+1$, and $n+2$ are all pretty good. Find the number of clever positive integers less than $10^{2018}$. Note: the only number divisible by 0 is 0.

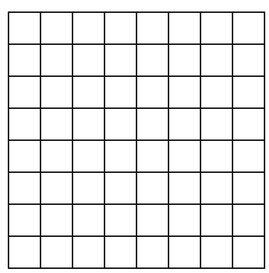
11. **[40]** What is the area in the $xy$-plane bounded by $x^2 + \frac{y^2}{3} \leq 1$ and $\frac{x^2}{3} + y^2 \leq 1$?

12. **[45]** Let $S$ be the set of ordered triples $(a, b, c) \in \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$. Let $n$ be the smallest positive integer such that there exists a polynomial, with integer coefficients, of the form
$$\sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} a_{(i,j,k)} x^i y^j z^k$$
such that the absolute value of all the coefficients are less than 2, and the polynomial equals 1 for all $(x, y, z) \in S$. Compute the number of such polynomials for that value of $n$.

13. **[45]** Let $N \geq 2017$ be an odd positive integer. Two players, A and B, play a game on an $N \times N$ board, taking turns placing numbers from the set $\{1, 2, \ldots, N^2\}$ into cells, so that each number appears in exactly one cell, and each cell contains exactly one number. Let the largest row sum be $M$, and the smallest row sum be $m$. A goes first, and seeks to maximize $\frac{M}{m}$, while B goes second and wishes to minimize $\frac{M}{m}$. There exists real numbers $a$ and $0 < x < y$ such that for all odd $N \geq 2017$, if A and B play optimally,
$$x \cdot N^a \leq \frac{M}{m} - 1 \leq y \cdot N^a.$$
Find $a$.

14. **[50]** Yunseo has a supercomputer, equipped with a function $F$ that takes in a polynomial $P(x)$ with integer coefficients, computes the polynomial $Q(x) = (P(x)-1)(P(x)-2)(P(x)-3)(P(x)-4)(P(x)-5)$, and outputs $Q(x)$. Thus, for example, if $P(x) = x+3$, then $F(P(x)) = (x+2)(x+1)(x)(x-1)(x-2) = x^5 - 5x^3 + 4x$. Yunseo, being clumsy, plugs in $P(x) = x$ and uses the function 2017 times, each time using the output as the new input, thus, in effect, calculating
$$\underbrace{F(F(F(\ldots F(F(}_{2017}x))\ldots))).$$
She gets a polynomial of degree $5^{2017}$. Compute the number of coefficients in the polynomial that are divisible by 5.

15. Fill in some of the squares of the $8 \times 8$ grid provided on the team round answer form so that it is impossible to place 10 or less rooks on the grid so that every empty square on the board either contains a rook or is being attacked by a rook. (Note that a rook is a chess piece that can move and attack any number of unoccupied squares horizontally and vertically and cannot move or attack through a filled-in square.) If your submission does not satisfy the condition (even if it's blank), then your team score will be multiplied by 0.9, but if your submission satisfies the condition and $N$ squares are filled in, then your team score will be multiplied by $0.01\lfloor 100.5 + \frac{512}{N^2} \rfloor$.



# Answers

## Individual Round

1. $12\pi - 16$
2. 8
3. 10
4. $(2, 2, 2, 2, 2, 2, 2, 2)$
5. 5050
6. 11
7. $\dfrac{-1+\sqrt{5}}{2}$
8. $2\pi - 3\sqrt{3}$
9. 8
10. $\dfrac{49}{256}$
11. 21600
12. 3027
13. $\dfrac{378}{5}$
14. $81_9$
15. 7
16. 25
17. 109, 180
18. $\dfrac{63}{10}$
19. 61
20. $3780\sqrt{5}$
21. $3\sqrt{2} - \sqrt{10}$
22. 38
23. $21°$
24. 2925
25. 2017

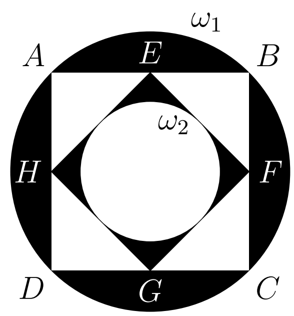
## Team Round

1. $75$
2. $7$
3. $-70$
4. $\frac{4023}{64}$
5. $3$
6. $4+2\sqrt[4]{8}$
7. $\dfrac{2}{7}$
8. $\dfrac{65}{8}$
9. $(4,3),(-3,-4),\left(\dfrac{-13-\sqrt{161}}{2},\dfrac{13-\sqrt{161}}{2}\right),\left(\dfrac{-13+\sqrt{161}}{2},\dfrac{13+\sqrt{161}}{2}\right)$
10. $1015$
11. $\dfrac{2\pi\sqrt{3}}{3}$
12. $941192$
13. $-1$
14. $5^{2017}-95$

# Solutions

## Individual Round

1. Square $ABCD$ is inscribed in circle $\omega_1$ of radius 4. Points $E$, $F$, $G$, and $H$ are the midpoints of sides $AB$, $BC$, $CD$, and $DA$, respectively, and circle $\omega_2$ is inscribed in square $EFGH$. The area inside $\omega_1$ but outside $ABCD$ is shaded, and the area inside $EFGH$ but outside $\omega_2$ is shaded. Compute the total area of the shaded regions.




*Proposed by Emily Wen*

**Solution:** Note that square $ABCD$ will have side length $4\sqrt{2}$, square $EFGH$ will have side length 4, and circle $\omega_2$ will have radius 2. Thus, the desired area is $(16\pi - 32) + (16 - 4\pi) = \boxed{12\pi - 16}$.

2. In a very strange orchestra, the ratio of violins to violas to cellos to basses is $4 : 5 : 3 : 2$. The volume of a single viola is twice that of a single bass, but $\frac{2}{3}$ that of a single violin. The combined volume of the cellos is equal to the combined volume of all the other instruments in the orchestra. How many basses are needed to match the volume of a single cello?

*Proposed by Ishika Shah*

**Solution:** Suppose that a bass has volume $x$. Then, a viola has volume $2x$, and a violin has volume $3x$. Using the third piece of information, we see that a cello has volume $\dfrac{4 \times 3x + 5 \times 2x + 2 \times x}{3} = 8x$, so we need $\boxed{8}$ basses to match the volume of one cello.

3. The 75 students at MOP are partitioned into 10 teams, all of distinct sizes. What is the greatest possible number of students in the fifth largest group? (A team may have only 1 person, but must have at least 1 person.)

*Proposed by Vincent Bian*

**Solution:** Note that if the 5th largest group has more than 10 people, then there would need to be at least $1+2+3+4+5+11+12+13+14+15 = 80$ people, contradiction. Thus, $\boxed{10}$ is the maximum, achieved when there are $1, 2, 3, 4, 5, 10, 11, 12, 13, 14$ people in the 10 groups, respectively.

4. Find all ordered 8-tuples of positive integers $(a, b, c, d, e, f, g, h)$ such that:

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{e^3} + \frac{1}{f^3} + \frac{1}{g^3} + \frac{1}{h^3} = 1.$$

*Proposed by Brandon Wang*

**Solution:** Observe that none of the integers can be 1, so $a \geq 2 \implies \frac{1}{a^3} \leq \frac{1}{8}$, where equality holds iff $a = 2$. We get similar inequalities for the other 7 variables. Summing, we get $\frac{1}{a^3} + \frac{1}{b^3} + \cdots + \frac{1}{h^3} \leq 1$; since equality must hold, we get our only solution, which is $\boxed{(2,2,2,2,2,2,2,2)}$.

5. Evaluate

$$\sum_{n=0}^{100} (-1)^n n^2.$$

*Proposed by Vincent Bian*

**Solution:** We can group the terms to get $(2^2 - 1^2) + (4^2 - 3^2) + \cdots + (100^2 - 99^2) = 3 + 7 + \cdots 199$. The average of these 50 terms is 101, so the total sum is $\boxed{5050}$.

6. Alan starts with the numbers 1, 2, ..., 10 written on a blackboard. He chooses three of the numbers, and erases the second largest one of them. He repeats this process until there are only two numbers left on the board. What are all possible values for the sum of these two numbers?

*Proposed by Ray Li*

**Solution:** Note that 1 and 10 will never be erased; thus there is only 1 possible value, which is $\boxed{11}$.

7. The first three terms of an infinite geometric series are 1, $-x$, and $x^2$. The sum of the series is $x$. Compute $x$.

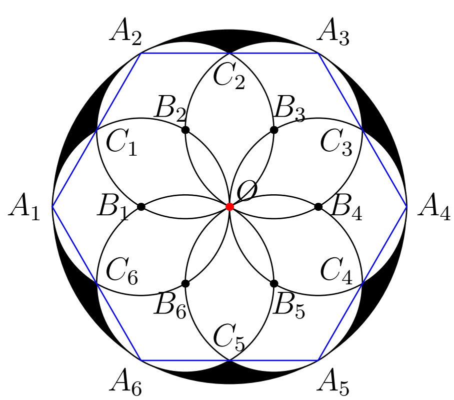
*Proposed by Ishika Shah*

**Solution:** Note that the sum is finite, so we have $|x| < 1$. Now, note that

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} = x,$$

so $x^2 + x - 1 = 0$. The roots of this polynomial are $\frac{-1 \pm \sqrt{5}}{2}$; since $|x| < 1$ we must have $\boxed{x = \frac{-1+\sqrt{5}}{2}}$.

8. Regular hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in a circle with center $O$ and radius 2. 6 circles with diameters $OA_i$ for $1 \leq i \leq 6$ are drawn. Compute the area inside the circle with radius 2 but outside the 6 smaller circles.




*Proposed by Jeffery Li*

**Solution:** Let $B_i$ be the midpoint of $OA_i$ and $C_i$ be the midpoint of $A_iA_{i+1}$. Thus, we can see that the circles we are dealing with are the circumcircles of $OC_iA_{i+1}C_{i+1}$ and have center $B_i$ (where $A_7 = A_1, C_7 = C_1$). Note that the area of the union of the 6 circles consists of the polygon $B_1C_1B_2C_2B_3C_3B_4C_4B_5C_5B_6C_6$ and 6 $120°$ arcs of circles of radius 1. The former is $\frac{1}{2}$ the area of $A_1A_2A_3A_4A_5A_6$ (since $[OB_iC_iB_{i+1}] = \frac{1}{2}[OA_iA_{i+1}]$ for all $1 \le i \le 6$) and thus has area $\frac{3\sqrt{3}\times 2^2}{2\times 2} = 3\sqrt{3}$, and the latter 6 arcs contribute $6 \times \frac{\pi}{3} = 2\pi$ to the area, so the area of the union of the 6 circles is $2\pi + 3\sqrt{3}$. Thus, since all 6 circles are contained in the circle of radius 2, the desired area is $4\pi - (2\pi + 3\sqrt{3}) = \boxed{2\pi - 3\sqrt{3}}$.

9. Let $a$, $b$, $c$ be distinct integers from the set $\{3, 5, 7, 8\}$. What is the greatest possible value of the units digit of $a^{(b^c)}$?

   *Proposed by Vincent Bian*

   **Solution:** Note that the last digit of $8^{5^3}$ is 8, so we only need to show that 9 is not achievable. To get 9, we would need 3 or 7 raised to a power that is 2 (mod 4). The only way to get an even power in the exponent is to use the 8, but that would make the exponent become 0 (mod 4). Thus, 9 is not achievable, and $\boxed{8}$ is the maximum.

10. I am playing a very curious computer game. Originally, the monitor displays the number 10, and I will win when the screen shows the number 0. Each time I press a button, the number onscreen will decrease by either 1 or 2 with equal probability, unless the number is 1, in which case it will always decrease to zero. What is the probability that it will take me exactly 8 button presses to win?

    *Proposed by Vincent Bian*

    **Solution:** Note that exactly 2 presses must decrease the onscreen number by 2, so there are $\binom{8}{2} = 28$ ways to win with 8 presses. Out of all $2^8 = 256$ possibilities for 8 button presses $\binom{7}{2} = 21$ ways end with the number decreasing by 1 and 7 ways end with the number decreasing by 2, so since the probability of the decreasing from 1 to 0 is $1 = 2 \times \frac{1}{2}$, we have to multiply the probability of ending with the number decreasing by 1 by a factor of 2, so the probability is $2 \times \frac{21}{256} + \frac{7}{256} = \boxed{\frac{49}{256}}$.

11. How many ways are there to rearrange the letters in "ishika shah" into two words so that the first word has six letters, the second word has four letters, and the first word has exactly five distinct letters? Note that letters can switch between words, and switching between two of the same letter does not produce a new arrangement.

    *Proposed by Eric Gan*

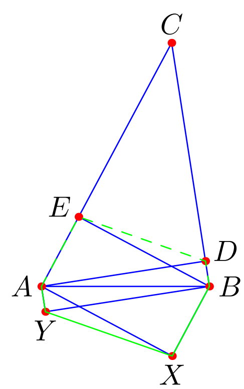
    **Solution:** Note that the letters that are repeated are "h" (3 times), "i", "s", and "a" (each 2 times), with "k" appearing only once. We want the first word to have exactly one letter repeated twice. If there are 2 "h"s in the first word, then no letter in the second word is repeated twice, so there are $4! \times \frac{6!}{2} = 8640$ ways in this case. Else, there will be 2 "h"s in the second word, so there are $3 \times \frac{4!}{2} \times \frac{6!}{2} = 12960$ ways in this case (we multiplied by 3 since either "i", "s", or "a" can be repeated twice in the first word). Thus, there are a total of $\boxed{21600}$ ways.

12. Let
$$N = 2018^{\frac{2018}{1+\frac{1}{\log_2(1009)}}}.$$
$N$ can be written in the form $a^b$, where $a$ and $b$ are positive integers and $a$ is as small as possible. What is $a+b$?

*Proposed by Ray Li*

**Solution:** Simplifying, we get
$$N = 2018^{2018/(\log_2(2018)/\log_2(1009))} = 2018^{2018\log_{2018}(1009)} = 1009^{2018};$$
Thus, $a+b = \boxed{3027}$.

13. Triangle $ABC$ has $AB = 17$, $BC = 25$, and $CA = 28$. Let $X$ be the point such that $AX \perp AC$ and $AC \parallel BX$. Let $Y$ be the point such that $BY \perp BC$ and $AY \parallel BC$. Find the area of $ABXY$.




*Proposed by Ishika Shah*

**Solution:** Let $D$ and $E$ be the feet of the altitudes from $A$ and $B$, and let $H$ be the orthocenter. Through angle and length chasing, we note that $ABXY$ is congruent to $BAED$. Now, note that we can split $ABC$ into a $8-15-17$ triangle and a $15-20-25$ triangle, so we note that $\sin(\angle DHE) = \sin(180 - \angle ABC) = \sin(\angle ABC) = \frac{3}{5}$, implying that $BE = BC\sin(\angle ABC) = 15$, $AD = AC\sin(\angle ABC) = \frac{84}{5}$, so
$$[BAED] = \frac{AD \times BE \times \sin(\angle DHE)}{2} = \frac{1}{2} \times 15 \times \frac{84}{5} \times \frac{3}{5} = \boxed{\frac{378}{5}}.$$

14. Find the last two digits of $17^{(20^{17})}$ in base 9. (Here, the numbers given are in base 10.)

*Proposed by Karen Ge*

**Solution:** It suffices to find $17^{20^{17}} \pmod{81}$, then convert this number (which is in base ten) into base nine. In order to do this, motivated by Euler's theorem, we find $20^{17}$ modulo $\varphi(81) = 54$. $20^{17} \equiv 0 \pmod 2$ and $20^{17} \equiv (-7)^{17} \equiv -7^{-1} \pmod{27}$, again by Euler's theorem since $\varphi(27) = 18$. But $7^{-1} \equiv 4 \pmod{27}$, so $20^{17} \equiv -4 \pmod{27}$. Now we have $20^{17} \equiv -4 \pmod{54}$ by the Chinese Remainder Theorem. Then the original number is equivalent to $17^{-4} \pmod{81}$. After some computation, we get that this number is $10^{-1} \equiv -8 \equiv 73 \pmod{81}$. Since $73_{10} = 81_9$, we have $\boxed{81_9}$ as our final answer.

15. How many positive integers $n$ satisfy the following properties:

 (a) There exists an integer $a > 2017$ such that $\gcd(a, n) = 2017$.

(b) There exists an integer $b < 1000$ such that $\operatorname{lcm}(b, n) = 2017000$.

*Proposed by Vincent Bian*

**Solution:** The first condition implies that $n$ is a multiple of 2017, and note that $2017 \mid n$ is sufficient, since we can take $a = n + 2017$. Let $n = 2017m$ for some positive integer $m$.

The second condition implies that $m \mid 1000$, since $n \mid 2017000$. Also, if $m$ is not a multiple of neither 8 nor 125, then we must have $b = 1000$, which is not allowed. If $m$ is, however, then we can choose either $b = 8$ or $b = 125$ to make the lcm equal to 2017000.

Thus, we only have $\boxed{7}$ possible values for $n$, which correspond to $m = 8, 40, 125, 200, 250, 500, 1000$.

16. Nathan is given the infinite series
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$
Starting from the leftmost term, Nathan will follow this procedure:

(a) If removing the term does not cause the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will remove the term and move on to the next term.

(b) If removing the term causes the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will not remove the term and move on to the next term.

How many of the first 100 terms will he end up removing?

*Proposed by Jeffery Li*

**Solution:** Note that $\dfrac{13}{15} = 0.\overline{1101}_2$, and since
$$\frac{1}{2^k} = \sum_{i>0} \frac{1}{2^{k+i}},$$
we note that the only terms that Nathan can remove are the ones not in the unique binary representation of $\dfrac{13}{15}$, or the ones of the form $\dfrac{1}{2^{4k-1}}$ where $k$ is a positive integer (or else, by the above identity, the sum becomes too small). Thus, Nathan will remove exactly $\dfrac{1}{4}$ of the first 100 terms, or $\boxed{25}$ terms.

17. Find all positive integers $1 \le k \le 289$ such that $k^2 - 32$ is divisible by 289.

*Proposed by Nathan Ramesh*

**Solution:** First, note that if $k$ is a solution, then $289 - k$ is a solution. Also, $k^2 - 32 \equiv 0 \pmod{17}$ only has solutions $k \equiv 7, 10 \pmod{17}$, so we will focus on all solutions with $k \equiv 7 \pmod{17}$ and then find the other solutions by subtracting them from 289. Write $k = 17m + 7$ with $0 \le m \le 16$, so that $(17m + 7)^2 - 32 \equiv 14 \times 17m + 17 \equiv 0 \pmod{289}$. Solving, we get $m \equiv -14^{-1} \equiv 6 \pmod{17}$; since $m \le 16$ we must have $m = 6$. Thus, we get $k = 109$, and $289 - k = 180$, which gives us $\boxed{109, 180}$ as the only solutions.

18. Daniel has a broken four-function calculator, on which only the buttons 2, 3, 5, 6, $+$, $-$, $\times$, and $\div$ work. Daniel plays the following game: First, he will type in one of the digits with equal probability, then one of the four operators with equal probability, then one of the digits with equal probability, and then he will let the calculator evaluate the expression. What is the expected value of the result he gets?

*Proposed by Jeffery Li*

**Solution:** We do casework based on the operation Daniel randomly chooses. If he chooses addition, then the expected value is

$$\frac{2+3+5+6}{4}+\frac{2+3+5+6}{4}=8.$$

If he chooses subtraction, then the expected value is

$$\frac{2+3+5+6}{4}-\frac{2+3+5+6}{4}=0.$$

If he chooses multiplication, then the expected value is

$$\left(\frac{2+3+5+6}{4}\right)\left(\frac{2+3+5+6}{4}\right)=16.$$

If he chooses division, then the expected value is

$$\left(\frac{2+3+5+6}{4}\right)\left(\frac{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}}{4}\right)=1.2.$$

Thus, the expected value of the result is

$$\frac{8+0+16+1.2}{4}=\boxed{6.3}.$$

19. We call a pair of positive integers *quare* if their product is a perfect square. A subset $S$ of $\{1, 2, \dots, 100\}$ is chosen, such that no two distinct elements of $S$ form a quare pair. Find the maximum possible numbers in such a set $S$.

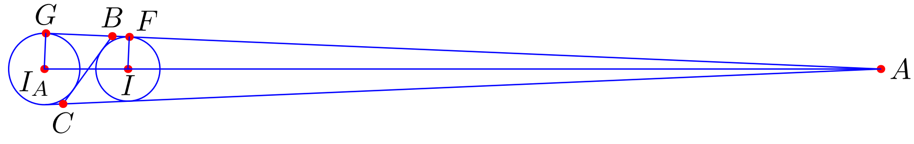
*Proposed by Nathan Ramesh*

**Solution:** Group the numbers into the following subsets:

$$\{1, 4, 9, 16, \dots, 100\}$$
$$\{2, 8, 18, 32, \dots, 98\}$$
$$\{3, 12, 27, 48, 75\}$$
$$\{5, 20, 45, 80\}$$
$$\dots$$
$$\{97\}$$

Note that each subset contains exactly one square-free integer and $S$ cannot contain more than one element from each subset, so it suffices to compute the number of square-free integers less than 100, which is $100-\left\lfloor\frac{100}{4}\right\rfloor-\left\lfloor\frac{100}{9}\right\rfloor+\left\lfloor\frac{100}{36}\right\rfloor-\left\lfloor\frac{100}{25}\right\rfloor+\left\lfloor\frac{100}{100}\right\rfloor-\left\lfloor\frac{100}{49}\right\rfloor=\boxed{61}$.

20. Two circles of radii 20 and 18 are an excircle and incircle of a triangle $\mathcal{T}$. If the distance between their centers is 47, compute the area of $\mathcal{T}$.

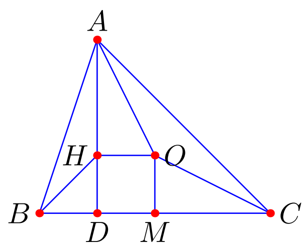



*Proposed by Nathan Ramesh*

**Solution:** Let $\mathcal{T} = \triangle ABC$, the incircle with radius 18 with center $I$ hit $AB$ at $F$, and the $A$-excircle with radius 20 with center $I_A$ hit $AB$ at $G$. Note that $FG = \sqrt{47^2 - 2^2} = 21\sqrt{5}$. Since $AIF$ and $AI_AG$ are similar, we get $\frac{AG - 21\sqrt{5}}{AG} = \frac{AF}{AG} = \frac{AI}{AI_A} = \frac{9}{10}$ so $AG = 210\sqrt{5}$. It's well-known that the length of $AG$ is equal to the semiperimeter of $ABC$; thus,

$$[\mathcal{T}] = rs = (18)(210\sqrt{5}) = \boxed{3780\sqrt{5}}.$$

21. Let $ABC$ be a triangle, $M$ be the midpoint of $BC$, and $D$ the foot of the altitude from $A$ to $BC$. Let $O$ be the circumcenter of $ABC$, and $H$ the orthocenter of $ABC$. If $OHDM$ is a square with side length 1, find $|AC - AB|$.




*Proposed by Eric Gan*

**Solution:** First, note that

$$AO = OC \implies AH^2 + HO^2 = AO^2 = OC^2 = OM^2 + MC^2$$

$$\implies AH = MC \implies AD = DC,$$

so $ADC$ is an isosceles right triangle. Thus, $\angle BCA = 45°$, so $\angle CBH = 45°$, so $BD = BH = 1$. Thus, $AH = CM = BM = 2$, so $AD = 3$, $DC = 3$, and $BD = 1$. Thus, by Pythagorean Theorem, $AB = \sqrt{3^2 + 1^2} = \sqrt{10}$, $AC = \sqrt{3^2 + 3^2} = 3\sqrt{2}$, so

$$|AC - AB| = \boxed{3\sqrt{2} - \sqrt{10}}.$$

*Note:* A similar problem appeared in the 2017 PUMaC Geometry A Round (#5), and on an old Putnam contest.

22. The value of the expression

$$\sqrt{1 + \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}} + \sqrt{1 - \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}}$$

can be written as $\sqrt[m]{n}$, where $m$ and $n$ are positive integers. Compute the smallest possible value of $m + n$.

*Proposed by Sam Ferguson*

**Solution:** Let $x$ be the value of the expression and let $a = \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}$. Then $x = \sqrt{1 + a} + \sqrt{1 - a}$, so $x^2 = 2 + 2\sqrt{1 - a^2}$. Substituting in the value for $a$ gives

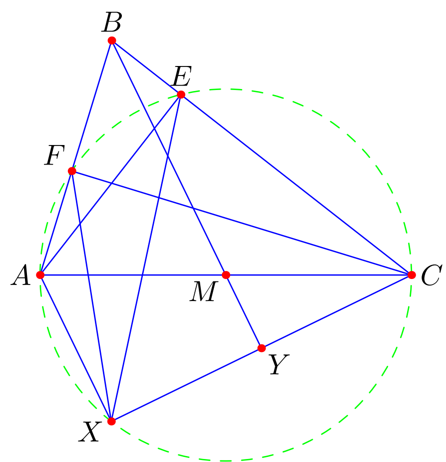
$$x^2 = 2 + 2\sqrt{1 + \sqrt[3]{16} - \sqrt[3]{32}}$$

$$\implies x^2 = 2 + 2\sqrt{\sqrt[3]{16} - 2\sqrt[3]{4} + 1}$$

$$\implies x^2 = 2 + 2(\sqrt[3]{4} - 1)$$
$$\implies x^2 = 2\sqrt[3]{4}$$
$$\implies x = \sqrt[6]{32}.$$

This is clearly the most simplified form, so $m + n = 6 + 32 = \boxed{38}$.

23. Triangle $ABC$ has $\angle A = 73°$, and $\angle C = 38°$. Let $M$ be the midpoint of $AC$, and let $X$ be the reflection of $C$ over $BM$. If $E$ and $F$ are the feet of the altitudes from $A$ and $C$, respectively, then what is the measure, in degrees, of $\angle EXF$?




*Proposed by Vincent Bian*

**Solution:** Let $Y$ be the foot of the perpendicular from $C$ to $BM$, so $CX = 2CY$. Also, $CA = 2CM$, so by SAS, $\triangle ACX \sim \triangle MCY$. Thus, because $CY \perp BM$, we have $\angle AXC = 90°$, so $A$, $X$, $C$, $E$, and $F$ lie on a circle. Since $\angle A > \angle C$, $X$ will lie on the opposite side of $AC$ as $B$, and

$$\angle EXF = \angle ECF = \angle BCF = 90° - \angle B = \angle A + \angle C - 90° = 73° + 38° - 90° = \boxed{21°}.$$

24. Compute the remainder when $\binom{3^4}{3^2}$ is divided by $3^8$.

*Proposed by Jeffery Li*

**Solution:** Consider $4480\left(\binom{81}{9} - 9\right)$, which is equal to $(x-1)(x-2)\dots(x-8) - 8!$ where $x = 81$. We only need to consider the linear coefficient (since the constant is 0 and any other coefficients are irrelevant since $x^2 = 3^8$ so $3^8 \mid x^i$ for $i > 1$), which is

$$-8!\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) = -8!\left(\frac{1}{2} + \frac{9}{8} + \frac{9}{14} + \frac{9}{20}\right) \equiv \frac{-8!}{2} \equiv 9 \pmod{81}$$

since $v_3(8!) = 2$. Thus, we have that

$$4480\left(\binom{81}{9} - 9\right) = (x-1)(x-2)\dots(x-8) - 8! \equiv 9x = 3^6 \pmod{3^8}$$
$$\implies \binom{81}{9} - 9 \equiv \frac{3^6}{4480} \equiv \frac{3^6}{7} \equiv 4 \times 3^6 = 2916 \pmod{3^8}$$

$$\implies \binom{81}{9} \equiv \boxed{2925} \pmod{3^8}.$$

(Note that the actual value is 260887834350, which does indeed leave a remainder of 2925 when divided by 6561.)

25. Consider sets $S = \{s_1, s_2, \dots, s_n\}$ of integers for which each of $\{1, 2, 4, 8, 16\}$ can be written as the sum of distinct elements of $S$. Find the minimum possible value of

$$500n + |s_1| + |s_2| + \cdots + |s_n|.$$

*Proposed by Evan Chen*

**Solution:** Let $A = 500n + |s_1| + |s_2| + \cdots + |s_n|$.

First, we contend that no such sets exist for $n \le 3$. Indeed, $S$ must contain at least one odd element. So then at most half the subsets of $S$ have even sum, and one of those even sums is zero. Hence we need $2^{n-1} \ge 5$, requiring $n \ge 4$.

We next claim that if $n = 4$ then $\sum |s_i| > 16$. Assume this is not the case. Since some subset of $S$ has sum equal to 16, the triangle inequality implies $s_1 + s_2 + s_3 + s_4 = 16$ and $s_i > 0$ for all $i$. If we assume that $1 \le s_1 < s_2 < s_3 < s_4$, this forces $s_1 = 1$, $s_2 = 2$. Thus we would need $s_3 \in \{3, 4\}$ but these lead to $s_4 \in \{9, 10\}$ and in this situation no subset of $S$ has sum 8.

In conclusion, we have proved $A \ge 500 \cdot 4 + 17 = \boxed{2017}$. An example is given by $S = \{-1, 2, 5, 9\}$.

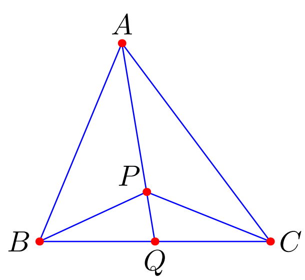
## Team Round

1. A team of 6 distinguishable students competed at a math competition. They each scored an integer amount of points, and the sum of their scores was 170. Their highest score was a 29, and their lowest score was a 27. How many possible ordered 6-tuples of scores could they have scored?

   *Proposed by Jeffery Li*

   **Solution:** First, note that the only possible ways that the students could've scored if they were indistinguishable were 29,29,29,28,28,27 or 29,29,29,29,27,27. In particular, note that 29,29,28,28,28,28 is not valid since the lowest score is not a 27. For the first case, there are $\frac{6!}{3!2!1!} = 60$ ways, and for the second case, there are $\frac{6!}{4!2!} = 15$ ways. Thus, there are a total of $\boxed{75}$ ways for the students to have scored in such a way.

2. Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let $P$ be a point inside triangle $ABC$, and let ray $AP$ meet segment $BC$ at $Q$. Suppose the area of triangle $ABP$ is three times the area of triangle $CPQ$, and the area of triangle $ACP$ is three times the area of triangle $BPQ$. Compute the length of $BQ$.




   *Proposed by Brandon Wang*

   **Solution:** Let the angle between $AQ$ and $BC$ be $\theta < 90°$. Then, note that $[ABP] = \frac{AP \times BQ \sin\theta}{2}$, $[ACP] = \frac{AP \times CQ \sin\theta}{2}$, $[BPQ] = \frac{PQ \times BQ \sin\theta}{2}$, and $[CPQ] = \frac{PQ \times CQ \sin\theta}{2}$. Now, we have

$$[ABP] = 3[CPQ], [ACP] = 3[BPQ]$$

$$\implies AP \times BQ = 3PQ \times CQ, AP \times CQ = 3PQ \times BQ$$

$$\implies \frac{BQ}{CQ} = \frac{CQ}{BQ} = \frac{3PQ}{AP}$$

$$\implies BQ^2 = CQ^2 \implies BQ = CQ$$

$$\implies BQ = \frac{BC}{2} = \boxed{7}.$$

3. I am thinking of a geometric sequence with 9600 terms, $a_1$, $a_2$,..., $a_{9600}$. The sum of the terms with indices divisible by three (i.e. $a_3 + a_6 + \cdots + a_{9600}$) is $\frac{1}{56}$ times the sum of the other terms (i.e. $a_1 + a_2 + a_4 + a_5 + \cdots + a_{9598} + a_{9599}$). Given that the terms with even indices sum to 10, what is the smallest possible sum of the whole sequence?

   *Proposed by Vincent Bian*

**Solution:** If the common ratio is $\frac{1}{r}$, then the ratio of the terms whose indices is divisible by 3 to the rest of the terms is $\frac{1}{r+r^2}$, so $r+r^2=56$, meaning $r$ is either 7 or $-8$.

Note that if the even indices add to 10, then the odd indices must add to $10r$, so the smallest possible sum is $10-8*10=\boxed{-70}$.

4. Let $ABCD$ be a regular tetrahedron with side length $6\sqrt{2}$. There is a sphere centered at each of the four vertices, with the radii of the four spheres forming a geometric series with common ratio 2 when arranged in increasing order. If the volume inside the tetrahedron but outside the second largest sphere is 71, what is the volume inside the tetrahedron but outside all four of the spheres?

*Proposed by Sruthi Parthasarathi*

**Solution:** Note that the volume of $ABCD$ is $\frac{s^3\sqrt{2}}{12}=72$. Thus, the volume of the region inside both the tetrahedron and the second largest sphere is $72-71=1$. Since the radii of the sphere form a geometric series with common ratio $\frac{1}{2}$, their volumes form a geometric series with common ratio $\frac{1}{8}$, so the volume common to the tetrahedron and the four spheres are $8, 1, \frac{1}{8}$, and $\frac{1}{64}$. Also, note that none of the spheres have radii greater than $3\sqrt{2}$, or else the overlapping region between that sphere and the tetrahedron contains a tetrahedron of side length $3\sqrt{2}$ and thus that volume is greater than 9, contradiction. Thus, since none of the spheres have radius greater than $3\sqrt{2}$, they don't overlap with each other, so the desired volume is $72-8-1-\frac{1}{8}-\frac{1}{64}=\boxed{\frac{4023}{64}}$.

5. Find the largest number of consecutive positive integers, each of which has exactly 4 positive divisors.

*Proposed by Carl Schildkraut*

**Solution:** Since 33, 34, and 35 each have exactly 4 positive integer divisors, the answer is at least 3. Now, assume for the sake of contradiction that the answer is at least 4. Then, there exists one positive integer divisible by 4 in our list. However, any number $4n$ has the divisors 1,2,4,$2n$, and $4n$, all of which are distinct unless $n=1$ or 2 (corresponding to 4 and 8). However, 4 only has 3 positive integer divisors, and 8 is not part of a list of 4 consecutive integers that each have 4 positive integer divisors (as 7 has only 2 and 9 has only 3), so the maximum number of these is $\boxed{3}$.

6. Let the (not necessarily distinct) roots of the equation $x^{12}-3x^4+2=0$ be $a_1, a_2, \ldots, a_{12}$. Compute

$$\sum_{i=1}^{12}|\mathrm{Re}(a_i)|.$$

*Proposed by Jeffery Li*

**Solution:** Note that the LHS factors as $(x-1)^2(x+1)^2(x^2+1)^2(x^4+2)$. Thus, the roots are $1, 1, -1, -1, i, i, -i, -i$, and $\frac{\pm\sqrt[4]{2}\pm i\sqrt[4]{2}}{\sqrt{2}}=\pm\frac{\sqrt[4]{8}}{2}\pm i\frac{\sqrt[4]{8}}{2}$. Thus, the sum is equal to $1+1+1+1+4\times\frac{\sqrt[4]{8}}{2}=\boxed{4+2\sqrt[4]{8}}$.

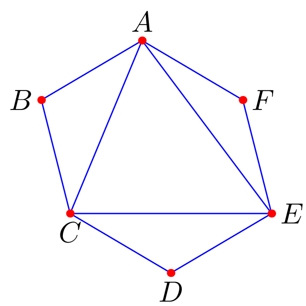
7. Jeffrey is doing a three-step card trick with a row of seven cards labeled A through G. Before he starts his trick, he picks a random permutation of the cards. During each step of his trick, he rearranges the cards in the order of that permutation. For example, for the permutation $(1, 3, 5, 2, 4, 7, 6)$, the first card from the left remains in position, the second card is moved to the third position, the third card

is moved to the fifth position, etc. After Jeffrey completes all three steps, what is the probability that the "A" card will be in the same position as where it started?

*Proposed by Eric K. Zhang*

**Solution:** Treat the permutation as a 1-regular functional graph. Then the problem amounts to finding the probability that "A" is in a cycle of length 1 or 3. The probability it is in a cycle of length one (a loop) is just $\frac{1}{7}$. In the case of a cycle of length three, simply multiply probabilities to get $\left(\frac{6}{7}\right)\left(\frac{5}{6}\right)\left(\frac{1}{5}\right) = \frac{1}{7}$, and add to get $\boxed{\frac{2}{7}}$.

8. In convex equilateral hexagon $ABCDEF$, $AC = 13$, $CE = 14$, and $EA = 15$. It is given that the area of $ABCDEF$ is twice the area of triangle $ACE$. Compute $AB$.




*Proposed by Jeffery Li*

**Solution:** Let $R$ be the circumradius of $ACE$. Reflect $B$ over $AC$ to $B'$, $D$ over $CE$ to $D'$, and $F$ over $AE$ to $F'$. Note that we want $[AB'C] + [CD'E] + [EF'A] = [ACE]$. However, as $AB$ increases, so does $[AB'C] + [CD'E] + [EF'A]$, and when $AB = R$, we will have $B' = D' = F' = O$, where $O$ is the circumcenter of $ACE$, giving equality. Thus, we must have $AB = R$, and through standard calculations (such as $R = \frac{abc}{4[ABC]}$), we get $AB = \boxed{\frac{65}{8}}$.

9. Find all ordered pairs $(x, y)$ of numbers satisfying

$$(1 + x^2)(1 + y^2) = 170$$
$$(1 + x)(-1 + y) = 10.$$

*Proposed by Eric Gan*

**Solution:** Note that the system of equations can be rewritten as:

$$(y - x)^2 + (1 + xy)^2 = 170$$
$$(y - x) + (1 + xy) = 12.$$

Thus, we have

$$\begin{aligned} &(y - x)^2 + 2(y - x)(1 + xy) + (1 + xy)^2 = 144 \\ \implies &(y - x)^2 - 2(y - x)(1 + xy) + (1 + xy)^2 = 196 \\ \implies &(y - x) - (1 + xy) = \pm 14 \\ \implies &y - x = 13, xy = -2 \text{ or } y - x = -1, xy = 12. \end{aligned}$$

Solving the first case gives $(x, y) = \boxed{\left(\frac{-13-\sqrt{161}}{2}, \frac{13-\sqrt{161}}{2}\right), \left(\frac{-13+\sqrt{161}}{2}, \frac{13+\sqrt{161}}{2}\right)}$; solving the second case gives $(x, y) = \boxed{(4,3), (-3,-4)}$. It's easy to check all of these work.

10. Call a positive integer "pretty good" if it is divisible by the product of its digits. Call a positive integer $n$ "clever" if $n$, $n+1$, and $n+2$ are all pretty good. Find the number of clever positive integers less than $10^{2018}$. Note: the only number divisible by 0 is 0.

    *Proposed by Sam Ferguson*

    **Solution:** There are exactly 7 single-digit clever integers. It's easy to see that no integer containing a digit 0 is pretty good. Furthermore, any clever integer greater than 9 must have all digits preceding the units digit be 1(s), as for any digit $d \neq 1$, at least one of $n, n+1, n+2$ is not divisible by $d$. Thus, all clever integers greater than 9 are of the form

$$\underbrace{111\cdots111}_{m\text{ 1s}}d$$

    for some integer $1 \leq m \leq 2017$ and some digit $d$. Since neither $\underbrace{111\cdots111}_{m\text{ 1s}}4$ nor $\underbrace{111\cdots111}_{m\text{ 1s}}8$ are pretty good for any integer $m \geq 1$, clever integers greater than 9 must be of form A: $n = \underbrace{111\cdots111}_{m\text{ 1s}}1$ or form B: $n = \underbrace{111\cdots111}_{m\text{ 1s}}5$. It's easy to see that an integer of form A is clever iff $m = 3k$ for some positive integer $k$, so there are $\left\lfloor \frac{2017}{3} \right\rfloor = 672$ clever integers of form A less than $10^{2018}$.

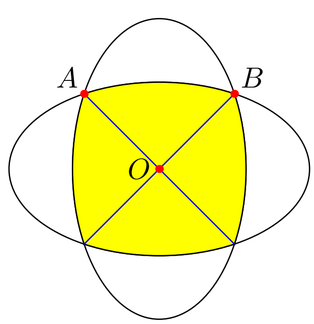
    For form B, clearly $n$ itself is divisible by 5, and $n+1$ is divisible by 6 iff $m = 3k$ for some positive integer $k$. To test whether $n+2$ is divisible by 7, we rewrite it as $n+2 = \frac{10^{m+1}-1}{9} + 6$. Then $7 \mid n+2$ iff

$$\frac{10^{m+1}-1}{9} \equiv 1 \mod 7 \iff 10^{m+1} \equiv 10 \mod 7 \iff m \equiv 0 \mod 6$$

    since $10^6 \equiv 1 \mod 7$ but $10^x \not\equiv 1 \mod 7$ for $1 \leq x \leq 5$. Thus an integer of form $B$ is clever iff $m = 6k$ for some positive integer $k$, so there are $\left\lfloor \frac{2017}{6} \right\rfloor = 336$ clever integers of form B less than $10^{2018}$.

    Thus, there are a total of $7 + 672 + 336 = \boxed{1015}$ clever integers less than $10^{2018}$.

11. What is the area in the $xy$-plane bounded by $x^2 + \frac{y^2}{3} \leq 1$ and $\frac{x^2}{3} + y^2 \leq 1$?




    *Proposed by Vincent Bian*

**Solution:** Note that the ellipses intersect at $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}\right)$, so let point $A$ be $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, let $B$ be $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, and let $O$ be the origin. Then, the desired area is made of 4 copies of the elliptical sector $AOB$ (where $AOB$ is a sector of the "horizontal" ellipse defined by $\frac{x^2}{3} + y^2 = 1$)

Now, consider the affine transformation $(x, y) \mapsto (\frac{x}{\sqrt{3}}, y)$, which sends the ellipse $\frac{x^2}{3} + y^2 = 1$ to the unit circle, so $AOB$ gets mapped to a circular sector $A'OB'$. Since $A'$ is at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B'$ is at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, we have that $\triangle A'OB'$ is equilateral. Thus, sector $A'OB'$ has one sixth of the area of a unit circle, so it has area $\frac{\pi}{6}$.

Since sector $A'OB'$ is $\frac{1}{\sqrt{3}}$ the area of sector $AOB$, sector $AOB$ has area $\frac{\pi\sqrt{3}}{6}$, and the entire intersection of the ellipses has area $\boxed{\frac{2\pi\sqrt{3}}{3}}$.

12. Let $S$ be the set of ordered triples $(a, b, c) \in \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$. Let $n$ be the smallest positive integer such that there exists a polynomial, with integer coefficients, of the form

$$\sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} a_{(i,j,k)} x^i y^j z^k$$

such that the absolute value of all the coefficients are less than 2, and the polynomial equals 1 for all $(x, y, z) \in S$. Compute the number of such polynomials for that value of $n$.

*Proposed by Jeffery Li*

**Solution:** We will first prove that $n = 6$. First, write the polynomial in terms of $x$, so that we get something of the form

$$\sum_{i=0}^{n} P_i(y, z) x^i$$

where $P_i(y, z)$ are polynomials in terms of $y, z$. Plugging in $-1$ and $1$ for $x$ gives us that

$$\sum_{1\leq 2j+1\leq n} P_{2j+1}(y, z) = 0$$

for all relevant ordered pairs $(y, z)$, so we can WLOG assume that no terms of the polynomial have an odd power of $x$ for now. We do something similar for $y$ and $z$, thus immediately getting that $n$ is even. Now, plugging in $(1, 0, 0)$ and permutations give that the coefficients of $x^n, y^n, z^n$ are all 1. Plugging in $(1, 1, 0)$ and permutations give that

$$\sum_{j=1}^{\frac{n}{2}-1} a_{(2j,n-2j,0)} x^{2j} y^{n-2j} = \sum_{j=1}^{\frac{n}{2}-1} a_{(0,2j,n-2j)} y^{2j} z^{n-2j} = \sum_{j=1}^{\frac{n}{2}-1} a_{(2j,0,n-2j)} x^{2j} z^{n-2j} = -1,$$

and plugging in $(1,1,1)$ gives that

$$\sum_{\substack{i+j+k=\frac{n}{2}\\ i,j,k>0}} a_{(2i,2j,2k)}x^{2i}y^{2j}z^{2k} = 1.$$

Thus, we must have at least one term of the form $x^{2i}y^{2j}z^{2k}$ $(i,j,k>0)$ that has a nonzero coefficient, so $n \geq 2(1+1+1) = 6$. At least one exists with $n=6$; take $x^6+y^6+z^6-x^4y^2-y^4z^2-z^4x^2+x^2y^2z^2$. Thus, we have that $n=6$.

Now, we find the number of such polynomials. From earlier, we must have the coefficients of $x^6, y^6, z^6$ be 1. Now, consider the terms that only contain $x$ and $y$, which are $x^iy^{6-i}$ for $1 \leq i \leq 5$. From plugging in $(1,1,0)$ and $(1,-1,0)$, we get that

$$\sum_{i=1}^{5} a_{(i,6-i,0)} = \sum_{i=1}^{5}(-1)^i a_{(i,6-i,0)} = 1$$

$$\implies a_{(1,5,0)} + a_{(3,3,0)} + a_{(5,1,0)} = 0, \quad a_{(2,4,0)} + a_{(4,2,0)} = 1.$$

Since all the terms have absolute value less than 2, there are 7 possible ways to choose the first 3 terms ($(0,0,0),(1,-1,0)$ and permutations) and 2 possible ways to choose the last 2 terms ($(1,0)$ and $(0,1)$). Thus, there are 14 ways to choose the terms $a_{(i,6-i,0)}$. Similarly, there are 14 ways to choose the terms $a_{(0,i,6-i)}$, and 14 ways to choose the terms $a_{(i,0,6-i)}$.

Now, consider the terms that contain $x$, $y$, and $z$. Denote $A_{xy} = a_{(1,1,4)} + a_{(1,3,2)} + a_{(3,1,2)}$, $A_{xz} = a_{(1,4,1)} + a_{(1,2,3)} + a_{(3,2,1)}$, and $A_{yz} = a_{(4,1,1)} + a_{(2,1,3)} + a_{(2,3,1)}$. Plugging in $(1,1,1), (-1,1,1), (1,-1,1)$, and $(1,1,-1)$, we get

$$a_{(2,2,2)} + A_{xy} + A_{xz} + A_{yz} = a_{(2,2,2)} - A_{xy} - A_{xz} + A_{yz}$$
$$= a_{(2,2,2)} - A_{xy} + A_{xz} - A_{yz} = a_{(2,2,2)} + A_{xy} - A_{xz} - A_{yz} = 1$$
$$\implies a_{(2,2,2)} = 1, A_{xy} = A_{yz} = A_{xz} = 0.$$

This gives us another $7^3$ ways to choose the coefficients (We can have $(a_{(1,1,4)}, a_{(1,3,2)}, a_{(3,1,2)})$ equal $(0,0,0)$, $(1,0,-1)$ and permutations, and similar for the terms in $A_{xz}$ and $A_{yz}$).

Thus, the total number of such polynomials is $14 \times 14 \times 14 \times 7^3 = 98^3 = \boxed{941192}$.

13. Let $N \geq 2017$ be an odd positive integer. Two players, A and B, play a game on an $N \times N$ board, taking turns placing numbers from the set $\{1, 2, \ldots, N^2\}$ into cells, so that each number appears in exactly one cell, and each cell contains exactly one number. Let the largest row sum be $M$, and the smallest row sum be $m$. A goes first, and seeks to maximize $\frac{M}{m}$, while B goes second and wishes to minimize $\frac{M}{m}$. There exists real numbers $a$ and $0 < x < y$ such that for all odd $N \geq 2017$, if A and B play optimally,

$$x \cdot N^a \leq \frac{M}{m} - 1 \leq y \cdot N^a.$$

Find $a$.

*Proposed by Brandon Wang*

**Solution:** Let $N = 2k+1$. First, let A place down $\dfrac{N^2+1}{2}$, and then let him do "strategy stealing;" i.e. if there's enough space, then if B places $x$ then let A place $N^2+1-x$ in the same row. This will guarantee that the row sums are between $k(N^2+1)+1$ and $k(N^2+1)+N^2$; thus, since $k = \dfrac{N-1}{2}$, this gives an upper bound of

$$\frac{(N-1)(N^2+1)+2N^2}{(N-1)(N^2+1)+2} = 1 + \frac{2N^2-2}{N^3-N^2+N+1} < 1 + \frac{2}{N-1} < 1 + \frac{c_1}{N}$$

for some constant $c_1 > 0$ and all $N \geq 2017$. B can force one of the row sums to be greater than $k(N^2+1)+N^2-k$ by placing $\frac{N+1}{2}$ of the largest numbers in the same row and making A place $\frac{N-1}{2}$ of the smallest numbers in the same row; since $m \leq \frac{1+2+\cdots+N^2}{N} = \frac{N^3+N}{2}$ we can get a lower bound of

$$\frac{k(N^2+1)+N^2-k}{\frac{N^3+N}{2}} = \frac{N^3+N^2}{N^3+N} = 1+\frac{N-1}{N^2+1} > 1+\frac{c_2}{N}$$

for some $c_2 > 0$, $c_2 < c_1$, and all $N \geq 2017$ Thus, $a = \boxed{-1}$.

14. Yunseo has a supercomputer, equipped with a function $F$ that takes in a polynomial $P(x)$ with integer coefficients, computes the polynomial $Q(x) = (P(x)-1)(P(x)-2)(P(x)-3)(P(x)-4)(P(x)-5)$, and outputs $Q(x)$. Thus, for example, if $P(x) = x+3$, then $F(P(x)) = (x+2)(x+1)(x)(x-1)(x-2) = x^5 - 5x^3 + 4x$. Yunseo, being clumsy, plugs in $P(x) = x$ and uses the function 2017 times, each time using the output as the new input, thus, in effect, calculating

$$\underbrace{F(F(F(\dots F(F(}_{2017}x))\dots))).$$

She gets a polynomial of degree $5^{2017}$. Compute the number of coefficients in the polynomial that are divisible by 5.

*Proposed by Jeffery Li*

**Solution:** We work in $\mathbb{F}_5[x]$, or the ring/set of polynomials with coefficients elements of $\mathbb{F}_5$, or integers mod 5. Note that, letting $P$ be shorthand for $P(x)$,

$$F(P) = (P-1)(P-2)(P-3)(P-4)(P-5) = P(P-1)(P-2)(P-3)(P-4) = P^5 - P$$

upon expansion.

Now, define the mapping $\Psi : \mathbb{F}_5[x] \to \mathbb{F}_5[x]$ such that

$$\Psi\left(\sum_{i=1}^{n} a_i x^i\right) = \sum_{i=1}^{n} a_i x^{5^i}.$$

It's easy to see that $\Psi(1) = x$ and $\Psi$ is additive; that is, $\Psi(P+Q) = \Psi(P) + \Psi(Q)$ where $P, Q \in \mathbb{F}_5[x]$. We also have that, if $P(x) = \sum_{i=1}^{n} a_i x^i$, then

$$\Psi(P)^5 = \left(\sum_{i=1}^{n} a_i x^{5^i}\right)^5 = \sum_{i=1}^{n} a_i x^{5^{i+1}} = \Psi(xP)$$

by the Frobenius Endomorphism. Now, we introduce the following:

Lemma: $F^n(x) = \Psi\left((x-1)^n\right)$.

Proof: We use proof by induction. The base case ($n=0$) is trivial, as $x = \Psi(1)$ is true. Now, suppose that $F^n(x) = \Psi\left((x-1)^n\right)$ for all $n \leq k$. Then,

$$\begin{aligned}
F^{k+1}(x) &= F(\Psi\left((x-1)^k\right)) \\
&= \Psi\left((x-1)^k\right)^5 - \Psi\left((x-1)^k\right) \\
&= \Psi\left(x(x-1)^k\right) - \Psi\left((x-1)^k\right) \\
&= \Psi\left((x-1)^{k+1}\right),
\end{aligned}$$

completing the inductive step.

Thus, we have that $F^{2017}(x) = \Psi\left((x-1)^{2017}\right)$, so the only coefficients that are nonzero (and thus not divisible by 5) are the ones corresponding to terms of the form $x^{5^b}$ where $\binom{2017}{b} \not\equiv 0 \pmod 5$. We can count the number of such $b$ by using Lucas' Theorem, which gives us that, if $b = \overline{b_4b_3b_2b_1b_0}_5$ and since $2017 = \overline{31032}_5$, then

$$\binom{2017}{b} \equiv \binom{3}{b_4}\binom{1}{b_3}\binom{0}{b_2}\binom{3}{b_1}\binom{2}{b_0} \pmod 5;$$

thus, for $\binom{2017}{b} \not\equiv 0 \pmod 5$, we need $0 \le b_4 \le 3, 0 \le b_3 \le 1, b_2 = 0, 0 \le b_1 \le 3, 0 \le b_0 \le 2$, which gives us $4 \times 2 \times 1 \times 4 \times 3 = 96$ such $b$. Thus, since the rest of the coefficients are zero, we see that $(5^{2017}+1) - 96 = \boxed{5^{2017} - 95}$ coefficients are divisible by 5.

*NEMO 2017 was directed and run by Emily Wen, Ishika Shah, Mihir Singhal, Sruthi Parthasarathi, Jeffery Li, and Eric Gan. We would like to thank Brandon Wang, Carl Schildkraut, Colin Tang, Eric K. Zhang, Evan Chen, Karen Ge, Le Nguyen, Nathan Ramesh, Ray Li, Sam Ferguson, Vincent Bian and Yuru Niu for submitting problem proposals. We would also like to thank Brandon Wang, Evan Chen, Vincent Bian, and Nathan Ramesh for helping with problem selection, and Brandon Wang for LaTeX and Asymptote help. Lastly, we would like to congratulate the 174 participants and 46 teams from 32 different schools for participating in this rather difficult contest.*