

Team Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 14 problems that are to be completed in 30 minutes, along with a small minigame that will generate a multiplier for a team's team round score.
- To ensure that your answers to problems 1 to 14 are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- The minigame, or problem 15, will ask your team to fill in an 8×8 grid satisfying certain conditions, and will generate a multiplier for your team round score. Be sure to try out the minigame, as failure to meet the conditions (even if nothing is submitted) may decrease your team's team round score.
- Each of the 14 team round problems have a predetermined point value; your team's team round score will be the sum of the point values assigned to each question that is correctly answered, multiplied by the multiplier generated by the minigame. Excluding the multiplier, a perfect score (achieved through answering all 14 problems correctly) is **400 points**.
- Your team score will be a combination of your score on the team round and the scores of each individual member.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your team name, team members' names, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. [10] A team of 6 distinguishable students competed at a math competition. They each scored an integer amount of points, and the sum of their scores was 170. Their highest score was a 29, and their lowest score was a 27. How many possible ordered 6-tuples of scores could they have scored?
2. [10] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let P be a point inside triangle ABC , and let ray AP meet segment BC at Q . Suppose the area of triangle ABP is three times the area of triangle CPQ , and the area of triangle ACP is three times the area of triangle BPQ . Compute the length of BQ .
3. [10] I am thinking of a geometric sequence with 9600 terms, $a_1, a_2, \dots, a_{9600}$. The sum of the terms with indices divisible by three (i.e. $a_3 + a_6 + \dots + a_{9600}$) is $\frac{1}{56}$ times the sum of the other terms (i.e. $a_1 + a_2 + a_4 + a_5 + \dots + a_{9598} + a_{9599}$). Given that the terms with even indices sum to 10, what is the smallest possible sum of the whole sequence?
4. [15] Let $ABCD$ be a regular tetrahedron with side length $6\sqrt{2}$. There is a sphere centered at each of the four vertices, with the radii of the four spheres forming a geometric series with common ratio 2 when arranged in increasing order. If the volume inside the tetrahedron but outside the second largest sphere is 71, what is the volume inside the tetrahedron but outside all four of the spheres?
5. [20] Find the largest number of consecutive positive integers, each of which has exactly 4 positive divisors.
6. [25] Let the (not necessarily distinct) roots of the equation $x^{12} - 3x^4 + 2 = 0$ be a_1, a_2, \dots, a_{12} . Compute

$$\sum_{i=1}^{12} |\operatorname{Re}(a_i)|.$$

7. [25] Jeffrey is doing a three-step card trick with a row of seven cards labeled A through G. Before he starts his trick, he picks a random permutation of the cards. During each step of his trick, he rearranges the cards in the order of that permutation. For example, for the permutation (1, 3, 5, 2, 4, 7, 6), the first card from the left remains in position, the second card is moved to the third position, the third card is moved to the fifth position, etc. After Jeffrey completes all three steps, what is the probability that the "A" card will be in the same position as where it started?
8. [30] In convex equilateral hexagon $ABCDEF$, $AC = 13$, $CE = 14$, and $EA = 15$. It is given that the area of $ABCDEF$ is twice the area of triangle ACE . Compute AB .
9. [35] Find all ordered pairs (x, y) of numbers satisfying

$$(1 + x^2)(1 + y^2) = 170$$

$$(1 + x)(-1 + y) = 10.$$
10. [40] Call a positive integer "pretty good" if it is divisible by the product of its digits. Call a positive integer n "clever" if n , $n + 1$, and $n + 2$ are all pretty good. Find the number of clever positive integers less than 10^{2018} . Note: the only number divisible by 0 is 0.
11. [40] What is the area in the xy -plane bounded by $x^2 + \frac{y^2}{3} \leq 1$ and $\frac{x^2}{3} + y^2 \leq 1$?
12. [45] Let S be the set of ordered triples $(a, b, c) \in \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$. Let n be the smallest positive integer such that there exists a polynomial, with integer coefficients, of the form

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} a_{(i,j,k)} x^i y^j z^k$$

such that the absolute value of all the coefficients are less than 2, and the polynomial equals 1 for all $(x, y, z) \in S$. Compute the number of such polynomials for that value of n .

