Individual Round

- **DO NOT open this test until your proctor tells you to begin.**
- This portion of the contest consists of 25 problems that are to be completed in 60 minutes.
- To ensure that your answers are marked correct if they are indeed correct, be sure that your answers are *simplified* and *exact*. Carry out any reasonable calculations (unless the answer obtained is greater than 10^{10}). Write fractional answers in the form $\frac{a}{b}$ where a, b are expressions not containing any fractions. Any decimals must be exact; rounded answers will not receive credit. Any square factors inside square roots must be moved outside the radical.
- There is no partial credit or penalty for incorrect answers.
- Each question will be weighted after the contest window according to the percentage of correct answers, and your individual score will be the sum of the point values assigned to each question that is correctly answered. A perfect score (achieved through answering all 25 problems correctly) is **200 points**.
- No aids other than the following are permitted: scratch paper, graph paper, ruler, compass, protractor, writing utensils, and erasers. **No calculators or other electronic devices (including smartwatches) are permitted.**
- Please make sure to record your name, school, and all answers on your answer form. Only the responses on the answer forms will be graded.

1. Square $ABCD$ is inscribed in circle ω_1 of radius 4. Points E, F, G, and H are the midpoints of sides AB, BC, CD, and DA, respectively, and circle ω_2 is inscribed in square $EFGH$. The area inside ω_1 but outside $ABCD$ is shaded, and the area inside $EFGH$ but outside ω_2 is shaded. Compute the total area of the shaded regions.

2. In a very strange orchestra, the ratio of violins to violas to cellos to basses is $4:5:3:2$. The volume of a single viola is twice that of a single bass, but $\frac{2}{3}$ that of a single violin. The combined volume of the cellos is equal to the combined volume of all the other instruments in the orchestra. How many basses are needed to match the volume of a single cello?

3. The 75 students at MOP are partitioned into 10 teams, all of distinct sizes. What is the greatest possible number of students in the fifth largest group? (A team may have only 1 person, but must have at least 1 person.)

4. Find all ordered 8-tuples of positive integers (a, b, c, d, e, f, g, h) such that:
$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}+\frac{1}{e^3}+\frac{1}{f^3}+\frac{1}{g^3}+\frac{1}{h^3}=1.$$

5. Evaluate
$$\sum_{n=0}^{100}(-1)^n n^2.$$

6. Alan starts with the numbers 1, 2, ..., 10 written on a blackboard. He chooses three of the numbers, and erases the second largest one of them. He repeats this process until there are only two numbers left on the board. What are all possible values for the sum of these two numbers?

7. The first three terms of an infinite geometric series are 1, $-x$, and x^2. The sum of the series is x. Compute x.

8. Regular hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in a circle with center O and radius 2. 6 circles with diameters OA_i for $1 \le i \le 6$ are drawn. Compute the area inside the circle with radius 2 but outside the 6 smaller circles.

9. Let a, b, c be distinct integers from the set $\{3, 5, 7, 8\}$. What is the greatest possible value of the units digit of $a^{(b^c)}$?

10. I am playing a very curious computer game. Originally, the monitor displays the number 10, and I will win when the screen shows the number 0. Each time I press a button, the number onscreen will decrease by either 1 or 2 with equal probability, unless the number is 1, in which case it will always decrease to zero. What is the probability that it will take me exactly 8 button presses to win?

11. How many ways are there to rearrange the letters in "ishika shah" into two words so that the first word has six letters, the second word has four letters, and the first word has exactly five distinct letters? Note that letters can switch between words, and switching between two of the same letter does not produce a new arrangement.

12. Let
$$N = 2018^{\frac{2018}{1+\frac{1}{\log_2(1009)}}}.$$
N can be written in the form a^b, where a and b are positive integers and a is as small as possible. What is $a+b$?

13. Triangle ABC has $AB = 17$, $BC = 25$, and $CA = 28$. Let X be the point such that $AX \perp AC$ and $AC \parallel BX$. Let Y be the point such that $BY \perp BC$ and $AY \parallel BC$. Find the area of $ABXY$.

14. Find the last two digits of $17^{(20^{17})}$ in base 9. (Here, the numbers given are in base 10.)

15. How many positive integers n satisfy the following properties:

    (a) There exists an integer $a > 2017$ such that $\gcd(a, n) = 2017$.

    (b) There exists an integer $b < 1000$ such that $\text{lcm}(b, n) = 2017000$.

16. Nathan is given the infinite series
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$
Starting from the leftmost term, Nathan will follow this procedure:

    (a) If removing the term does not cause the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will remove the term and move on to the next term.

    (b) If removing the term causes the remaining terms to sum to a value less than $\frac{13}{15}$, then Nathan will not remove the term and move on to the next term.

    How many of the first 100 terms will he end up removing?

17. Find all positive integers $1 \leq k \leq 289$ such that $k^2 - 32$ is divisible by 289.

18. Daniel has a broken four-function calculator, on which only the buttons 2, 3, 5, 6, $+$, $-$, $\times$, and $\div$ work. Daniel plays the following game: First, he will type in one of the digits with equal probability, then one of the four operators with equal probability, then one of the digits with equal probability, and then he will let the calculator evaluate the expression. What is the expected value of the result he gets?

19. We call a pair of positive integers *quare* if their product is a perfect square. A subset S of $\{1, 2, \ldots, 100\}$ is chosen, such that no two distinct elements of S form a quare pair. Find the maximum possible numbers in such a set S.

20. Two circles of radii 20 and 18 are an excircle and incircle of a triangle $\mathcal{T}$. If the distance between their centers is 47, compute the area of $\mathcal{T}$.

21. Let ABC be a triangle, M be the midpoint of BC, and D the foot of the altitude from A to BC. Let O be the circumcenter of ABC, and H the orthocenter of ABC. If $OHDM$ is a square with side length 1, find $|AC - AB|$.

22. The value of the expression
$$\sqrt{1 + \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}} + \sqrt{1 - \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}}$$
can be written as $\sqrt[m]{n}$, where m and n are positive integers. Compute the smallest possible value of $m + n$.

23. Triangle ABC has $\angle A = 73^\circ$, and $\angle C = 38^\circ$. Let M be the midpoint of AC, and let X be the reflection of C over BM. If E and F are the feet of the altitudes from A and C, respectively, then what is the measure, in degrees, of $\angle EXF$?

24. Compute the remainder when $\binom{3^4}{3^2}$ is divided by 3^8.

25. Consider sets $S = \{s_1, s_2, \ldots, s_n\}$ of integers for which each of $\{1, 2, 4, 8, 16\}$ can be written as the sum of distinct elements of S. Find the minimum possible value of
$$500n + |s_1| + |s_2| + \cdots + |s_n|.$$